Aesthetics, Geometry and the Yumi

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I. Confluence of aesthetics and pragmatism

Figure 1 displays the enigmatic note from Hashimoto sensei, forwarded to me after the '19 seminar in SC. It is a brief review of the Golden Ratio. An attached memo says that it "refers to the proportions of the yumi." In the

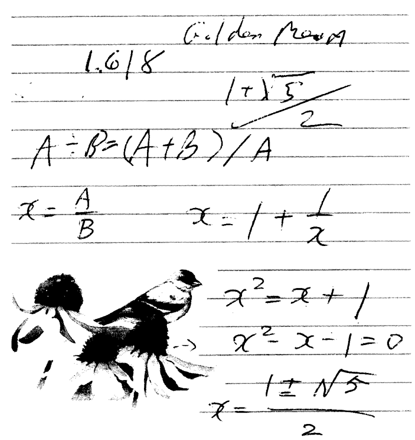

Figure 1: Hashimoto sensei's note

fifth century BC, the Greek sculptor and mathematician Phidias proposed the division of a line segment into two with "the most beautiful proportions." By retracing a path of feeling and intuition to that definite number, the Golden Ratio, we witness the arising of a *mathematical* form from pure *aesthetics*. Next, we see what it has to do with Kyudo. Let a and b denote lengths of the two segments, with a the greater, $a > b$. The division of a line into two defines not two, but *three* lengths, with the descending order,

$$\begin{aligned} a+b \qquad & \textit{(the total length)} \\ a \qquad & \textit{(the greater)} \\ b \qquad & \textit{(the lesser)} \end{aligned} \tag{1}$$

Comparison is often done by pairs, of which there are three:

$$\begin{aligned} a+b \quad & and \quad a \\ a \quad & and \quad b \\ a+b \quad & and \quad b \end{aligned} \tag{2}$$

The first two comparisons are the "closest" to each other, comparing one length to the next smaller. The intuitive sense of "most beautiful proportions" might arise like this: "We have two perspectives, of comparing the total to the greater, and the greater to the lesser. Let's impose *symmetry* as a certain equivalence between the two perspectives: The total stands in relation to the greater, the same as the greater to the lesser." The *numerical* comparison of two lengths is their ratio, so the quantitative expression is

$$\frac{a+b}{a} = \frac{a}{b}. \tag{3}$$

Defining x as the ratio $\frac{a}{b}$, this equation is equivalent to

$$1 + \frac{1}{x} = x, \tag{4}$$

whose *positive* solution is the Golden Ratio

$$x = \phi := \frac{1+\sqrt{5}}{2} \approx 1.61803... \tag{5}$$

Phidias originally applied the Golden Ratio to set proportions in human figure sculptures of the Parthenon temple. Once released into world culture, the Golden ratio strongly insinuates itself in artistic and architectural works over the two and a half millennia since Phidias. (Google "Golden Ratio" and you can see for yourself.)

Now, Kyudo! Let's examine the *proportion between yumi lengths above and below the top of the grip.* Denote these lengths a and b respectively. The

Table 1: Proportions of lengths above and below grip

a (cm)	b (cm)	$\frac{a+b}{a}$	$\frac{a}{b}$
144.0	88.7	1.616	1.623
144.0	88.7	1.618	1.618
144.0	88.5	1.619	1.627
143.8	89.0	1.619	1.616
142.7	90.2	1.631	1.582

first two columns of table 1 record measurements of a and b of five Yonsun yumi by the respected yumishi Don Symanski . The third and fourth columns are computed values of proportions $\frac{a+b}{a}$ and $\frac{a}{b}$.

The proportions in the last two columns are close to the Golden ratio. To obtain more refined "consensus" values, we calculate *geometric means*. For instance, the geometric mean of the values for $\frac{a+b}{a}$ in the third column is

$$\{(1.616)(1.618)(1.619)(1.619)(1.631)\}^{\frac{1}{5}} \approx 1.621. \tag{6}$$

The geometric mean (instead of the usual arithmetic mean) is most appropriate for *comparing proportions*. This is because the geometric mean of ratios is the ratio of geometric means. (The arithmetic mean of ratios is generally not equal to the ratio of arithmetic means.) Similarly, the geometric mean of the values for $\frac{a}{b}$ in the fourth column is 1.613.

I don't know that yumishi intentionally impose grip placement according to the Golden Ratio. Nevertheless, it is very nearly realized for the five yumi. Here is another example in Kyudo, very much in the original spirit of Phidias: On page 132 of the Kyudo Kyohan, there is an idealized line drawing of Kai. Superimposed upon it, there is the central vertical axis of the Kyudoka, and the three horizontal lines of shoulders, hips and feet, as in the Sanjumonji (three crosses). In the figure, the elevation of the shoulder line above the feet is 76.5 mm, and the elevation of the hip line, 47.5 mm. The ratio of elevations is 1.611.

Here, we will see that pragmatic features of yumi design related to the segmented character of bamboo strongly inform the proportions of lengths above and below the grip. These design features were *chosen* long ago and maintained by tradition over the centuries. They are very reliably present in all the five yumi's I've measured for this discourse. Figure 2 is a photograph of the first yumi in its braced configuration. There are six exposed belly nodes marked by black dots, and seven exposed back nodes marked by white dots. Certain observations turn out to be crucial: (i) The top of the grip is located at the fourth belly node from the top. It is labeled "grip" in figure 2. (ii) Each belly node is close to the halfway point between adjacent back nodes. (iii) The upper tip of yumi is just short of where the next back node would be. (iv) The lower tip of the yumi extends just beyond where the next belly node would be. On some of the five yumi, the belly bamboo continues under the lower strike plate, and the tell-tale of the "buried" belly node is a disturbance in the bamboo grain when you look from the side. There is typically another "buried" belly node underneath the upper strike plate.

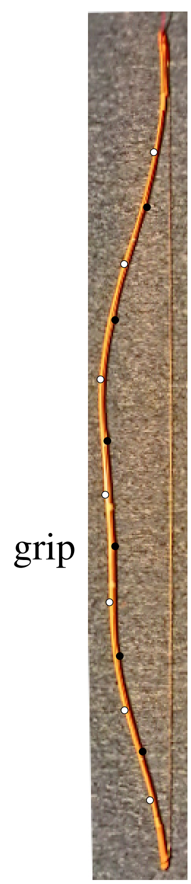

Figure 2: Nodes (back-white dots, belly-black dots)

Let's see what the observations (i)-(iv) have to say about proportions of lengths above and below the grip: By inspection of figure 2, we see that there are (slightly less than) four and one half inter-nodal intervals from the top of the grip to the upper tip, and (slightly more than) three below. Here, "inter-node intervals" refers to the intervals between nodes *on the same side*, belly or back. As a first approximation, assume that the spacings between bamboo nodes are uniform. With uniform node spacing, the approximation to the proportion between the lengths above and below the grip is

$$\frac{a}{b} \approx \frac{4.5}{3} = 1.5, \tag{7}$$

which is about 7.4% below the measured values in table 1. This shortfall reflects the *increase of inter-nodal spacings as we ascend the bamboo stalk*. We achieve the geometric mean value 1.613 of $\frac{a}{b}$ if the average inter-nodal spacing above the grip is larger than the spacing below by the factor $\frac{1.613}{1.5} \approx 1.08$.

The exercise just completed raises raises a possibility: The closeness of the yumi proportions $\frac{a+b}{a}$ and $\frac{a}{b}$ to the Golden Ratio emerges *naturally* from features of yumi design related to to segmented character of bamboo and the increase of inter-node spacings as you ascend the bamboo stalk.

II. Curve proportions

Imagine facing the yumi so you gaze along its back. In figure 2, your vantage point is somewhere well left of the grip. From this perspective, the top and bottom curves are concave. This appears to be a "natural" choice consistent with most archery traditions everywhere and throughout the ages. The curve containing the grip is concave as well, as if yumishi wanted to impart some extra Ikasu ("life") about the grip. The *mathematics* of curves dictates that concave and convex curves must alternate, so there must be at least one convex curve above the grip, and at least one other below. This makes five curves in all.

In mathematics, the contact between a concave curve and an adjacent convex curve is called an *inflection point*. The placing of inflection points is subtle and delicate: Small modifications of the yumi shape can drastically alter their positions. How are geometric proportions of all five curves in relation to each other determined? Is there an underlying aesthetics similar in spirit to Phidias' division of a line segment according to the Golden ratio?

The pragmatic side of the story begins with the unbraced shape of the yumi as initially created by the yumishi in the layup of the multiple lamina-

Figure 3: Shibata XXI presents the traditional ropes and splints layup of yumi.

tions (belly bamboo, back bamboo, and one or more core laminations). In traditional yumi art, the layup of the laminations is secured by ropes and splints. Figure 3 is a screen shot from a yumi making video by Shibata XXI. You can see the splints placed on the *convex* surface, belly or back, for each curve. What determines the intervals allotted to each of the five stacks of splints? I haven't had access to a yumishi who *tells* me what he does (so far), but by examining actual unbraced yumi, we can see the *results* quite independently of what may be said. Inspection of the five yumi suggests that the inflection points are close to alternating back and belly nodes. Figure 4 is a photograph of the first yumi, this time in its unbraced configuration. The nodes close to the inflection points are singled out by horizontal lines. Using these nodes as markers for inflection points of the unbraced yumi during layup *does* look like common sense. In any case, the marking of unbraced inflection points by nodes is very consistently upheld by the five yumi. Just as grip placement is strongly conditioned by the positions of back and belly nodes in traditional yumi design, so are the relative lengths of the five unbraced curves. By further inspection of figure 4, we surmise the number of internode intervals allotted to each curve, starting from the top. These are recorded in the second column of table 2. Under the naive assumption of uniform inter-nodal spacing, we can deduce the length of each curve as a fraction of total yumi length. These are listed in the second column of table 3.

Let's now concentrate on the *deviations* from these oversimplified proportions. For each of the five yumi, we can measure the lengths of the five curves, based upon the nominal inflection points as marked by belly or back nodes. For each curve we calculate the geometric mean of its measured fractional

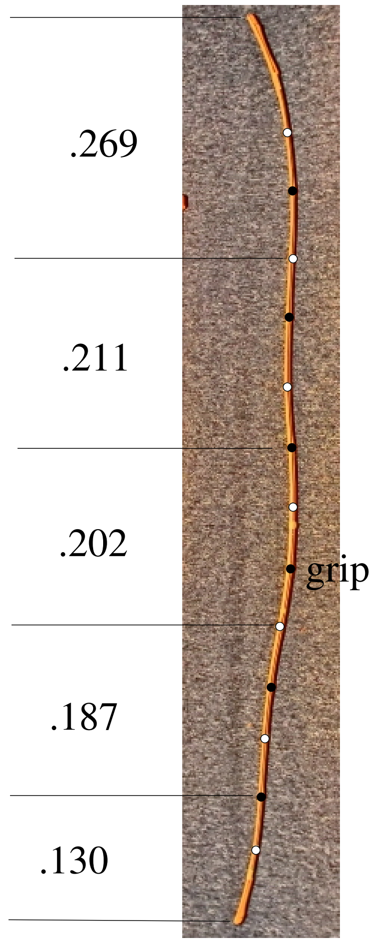

Figure 4: Inflection points and curve proportions of unbraced yumi

length. These are listed in the third column of table 3.

The "naive" curve proportions are not far from what is actually measured. This speaks for the *robustness* of traditional yumi design. Materials and yumishi vary, but the proportions which emerge are quite stable. The departures from naive values have their story as well: The progressive increase of observed fractional lengths as we ascend the three middle curves is a symptom of the inter-nodal spacings increasing as we ascend the bamboo stalk. The geometric means of fractional curve lengths from the third column of table 3 are posted in figure 4 for easy visual reference.

In contemplating the aesthetics of a yumi's shape, we have in mind *braced*

Table 2: Inter-nodal spacings in each curve

Curve	Inter-node spacings
1	(*slightly less than*) 2
2	3/2
3	3/2
4	3/2
5	(*slightly more than*) 1

Table 3: Fractional lengths of the curves

Curves	naive	geometric mean
1	.267	.269
2	.200	.211
3	.200	.202
4	.200	.187
5	.134	.130

yumi as we actually experience them in practice. The proportions between *depths* as well as lengths of curves must enter the discussion since both contribute to the yumi's shape. Bracing changes curve proportions: The curves which are convex when seen from the back become longer and deeper, the concave, shorter and shallower. There is a definite relationship between the unbraced and braced curves, but this is another discussion entirely.

Here, we look at braced curve proportions as they are, and what aesthetics might underlie them. From a certain mathematical averaging of measured curve proportions of many yumi, we can discern the shape representing a "consensus" among them. Ideally, we'd like access to a large ensemble of yumi by many yumishi, so the resulting shape would be a "broad consenus." Here, our consensus is more modest, deriving from the five Symanski Yonsunnobi. The consensus shape is a source of insight for aesthetic exploration. The "preservation of proportions" in the sense of Phidias has natural generalizations which set proportions of *all five curves in relation to each other*. The "consensus" shape definitely points to one of these proposals for harmonious curve proportions. The resulting yumi shape can be constructed mathematically. In summary: There is an emergent shape uniquely specified as a mathematical form which comes close indeed to a classic example of a "well proportioned yumi."

The actual work begins. How do we measure curve proportions in existing yumi? Since the back bamboo extends from tip to tip uninterrupted by strike plates and nocks, one edge of the back bamboo idealized as a plane curve is our proxy for the. yumi's shape. The top panel of figure 5 is the photograph of the first yumi, braced and reoriented horizontally. The black dot marks the belly node where the top of the grip would be. The hollow circles mark the ends of the yumi and inflection points in between. Notice that these circles straddle an edge of the back bamboo. The segments between these points are the five curves. The line segment connecting the endpoints of any given curve is called its *chord*. In the top panel of figure 5, we've drawn the the chord of the most prominent curve, second from the top. This curve is convex when seen from the back, and lies above its chord. The *elevation* h of a curve is the displacement of the curve's midpoint relative to its chord, reckoned *positive* for convex curves, *negative* for concave. Some practical details: The length of a curve is measured by a cloth tape measure hugging the back of the yumi between endpoints. The chord between endpoints is realized physically by a connecting thread. We measure the distance from the midpoint of the thread to the back of the yumi. This is the magnitude of the curve's elevation.

To gain an initial sense of curve proportions in existing yumi, we measure curve lengths and elevations of the five Symanski Yonsun-nobi in their *braced* configurations. The tsuru of each yumi is adjusted to produce the standard $15cm$ brace height (Ha) before measurements are taken. Having measured curve lengths and elevations of all five yumi, we seek the "consensus" of curve proportions. The proportions between lengths of different curves is straightforward: We convert measured lengths into fractions of total yumi length, and for each curve we compute the geometric mean of its fractional length. For Yonsun-nobi with its $233cm$ length, the corresponding sequence of physcial lengths from top to bottom is

$$58.7cm,\ 61.5cm,\ 35.9cm,\ 44.7cm,\ 31.2cm. \tag{8}$$

Similarly,, we compute the sequence of geometric mean curve elevations for Yonsun-nobi,

$$-3.2cm,\ 3.9cm,\ 0cm,\ 1.9cm,\ -1.0cm. \tag{9}$$

Elevation measurements are reproducible to within a millimeter, so the nonzero values are reported with two significant figures. Is the middle curve really flat with zero elevation? In practice, we find that it *is* very shallow in relation to the others. The second yumi has the deepest middle curve with an elevation of around $-4mm$, the first and third yumi, $-2mm$, $-3mm$, and for the last two yumi, flat to within a millimeter. The geometric mean of any data set containing zero vanishes. In summary, the depth of the middle curve is poorly resolved by direct measurement . We don't gain any sense of its proportion to the other curves save to say: "It is very much smaller." There *is* an indirect resolution of the middle curve. It involves the mathematical construction of the yumi shape from knowledge of its curve proportions.

Given a curve's length and elevation, its *spline* is a simple approximation to the whole curve shape modulo its position and orientation in the plane. The spline interpolates the endpoints and midpoint of the curve, and its curvature vanishes at the endpoints (as it must at inflection points). For instance, a sinusoidal spline is a half period of a sine wave between two adjacent inflection points. Given the lengths and elevations of all five curves (10 dimensions in all), we can construct their splines and join them end to end smoothly to produce an approximation to the shape of the yumi. The spline construction is easily carried out by computer and this is what we use to actually *see* the yumi shapes that emerge from proposed curve proportions. The green curve in the second panel of figure 5 is the approximate shape of the first yumi based on sinusoidal splines. Notice that the spline approximation hugs the the back of the yumi like it is supposed to, except for a small but discernible bulge in the second curve from the top. This particular yumi has a weak spot (which I knew about).

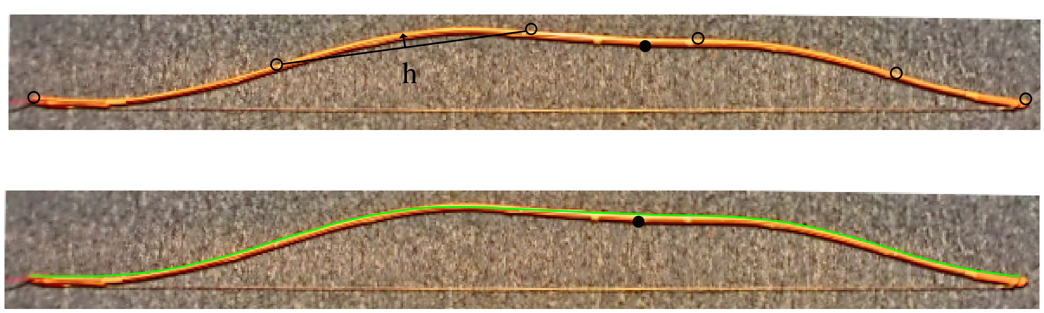

Figure 5: Spline approximation to the shape of a yumi

We construct the sinusoidal spline approximation to the "consensus" shape based upon the geometric means of curve lengths and elevations in (8), (9). This is the blue curve in figure 6. The horizontal line segment connects the top and bottom ends of this curve. At the top of the grip, the elevation of the blue curve relative to this "plumb line" is $16.4cm$. Too high: For each of the five yumi, extend a plumb line from one tip of the back bamboo to the other. At the grip, measure the elevation of the back bamboo relative to it. The average for the five yumi is $14.2cm$. The blue shape is too high because its middle curve is *flat*. The red curve is the shape resulting from the same lengths and elevations as in (8), (9), except that the elevation of the middle curve is $-2.4mm$. That's all you need to drop the elevation at the grip to $14.2cm$. In figure 7, the blue curves are the shapes of the five yumi: Photographs are imported into a graphics program and the blue curves are obtained by tracing along the back bamboo. The red curve from figure 6 is inserted among them. Three yumi follow the consensus shape well. One outlier is weak above the grip, the other strong.

Figure 6: Refinement of the consensus shape

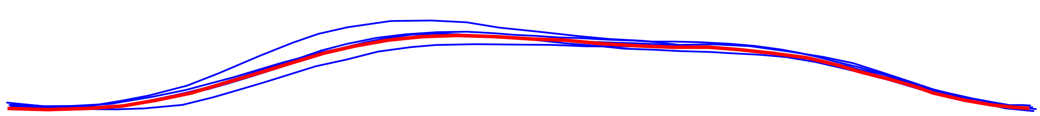

Figure 7: The blue curves are the shapes of the five individual yumi. The red curve is their "consensus," based on geometric means of curve lengths and elevations

Some reflections on the consensus shape informs the exploration of aesthetics: Unlike an unbraced yumi, the sequence of curve lengths (8) is not monotone increasing as we ascend the yumi from bottom to top. The bottom concave curve remains shorter than the convex curve above it. This is expected since bracing expands convex curves at the expense of the concave. The expansion of the convex curve above the grip is so large that it is now longer than the top concave curve. Qualitatively, the braced yumi has a kind of "mirror symmetry" of curve length proportions about the grip.

The length of the concave middle curve of an unbraced yumi is intermediate between the convex curves above and below. In figure 4, we see that is is quite deep. Bracing the yumi, it becomes shorter than its neighbor curves, but the most conspicuous feature is its relative *shallowness*. As we've seen, the elevation of the shape at the grip relative to the "plumb line" between its tips is very sensitive to the small depth of the middle curve. With these insights, where do we go from here?

We present two schemes of "preserving proportions" in the sense of Phidias which set the proportions of all five curves in relation to each other. In one proposal, the lengths of curves progressively increase by the *same* proportion R as we ascend from bottom to top. The sequence of fractional lengths is

$$\frac{1}{s}, \frac{R}{s}, \frac{R^2}{s}, \frac{R^3}{s}, \frac{R^4}{s}, \quad s := 1 + R + R^2 + R^3 + R^4. \tag{10}$$

There is a natural connection of R to the Golden Ratio: We assume that the top of the grip divides the whole yumi *and* the middle curve according to the Golden Ratio. The fractional length of the middle curve is $\frac{R^2}{s}$, and the fractional lengths of segments above and below the grip are

$$\frac{R^2}{s}\frac{\phi}{1+\phi} = \frac{R^2}{s\phi}, \tag{11}$$

and

$$\frac{R^2}{s\phi^2}. \tag{12}$$

In (11), we used $\frac{\phi}{1+\phi} = \frac{1}{\phi}$, in accord with equation (1) with $x = \phi$. Hence, the fractional lengths of yumi above and below the grip are

$$a = \frac{R^2}{s\phi} + \frac{R^3}{s} + \frac{R^4}{s}, \tag{13}$$

$$b = \frac{1}{s} + \frac{R}{s} + \frac{R^2}{s\phi^2}. \tag{14}$$

By requiring that the proportion of a and b is the Golden Ratio ϕ, we have the equation

$$\frac{R^2}{\phi} + R^3 + R^4 = \phi\left(1 + R + \frac{R^2}{\phi^2}\right). \tag{15}$$

Cancelling the common term $\frac{R^2}{\phi}$ from both sides leaves a reduced equation, each side of which has the common factor $1 + R$. All that is left is

$$R^3 = \phi, \tag{16}$$

so that R is the cube root of the Golden Ratio ϕ. The second column of table 4 lists the fractional curve lengths based on $R = \phi^{\frac{1}{3}} \approx 1.17398$. We call this sequence of length proportions "progressive."

Table 4: The "progressive" and "mirror symmetry" sequences of length proportions

Curve	progressive	mirror symmetry	consensus
1	.269	.213	.243
2	.229	.272	.271
3	.195	.213	.163
4	.166	.168	.195
5	.141	.132	.124

A second proposal retains the progressive increase of lengths from the bottom curve to the second curve from the top, but the top curve is shorter than the curve below in the *same* proportion as the bottom curve in relation to the curve above. This quantifies the "mirror symmetry" suggested by the consensus shape. The sequence of fractional lengths (10) is modified to

$$\frac{1}{s}, \frac{R}{s}, \frac{R^2}{s}, \frac{R^3}{s}, \frac{R^2}{s}, \quad s := 1 + R + R^2 + R^3 + R^2. \tag{17}$$

Here, the proportion of the top curve to the curve beneath it is $\frac{1}{R}$, the *same* as the proportion of the bottom curve to the curve above it. As before, the top of the grip divides the whole yumi and the middle curve according to the Golden Ratio. The expression for a in (13) modifies to

$$a = \frac{R^2}{s\phi} + \frac{R^3}{s} + \frac{R^2}{s}, \tag{18}$$

and the expression for b in (14) is unchanged. Equation (15) modifies to

$$\frac{R^2}{\phi} + R^3 + R^2 = \phi\left(1 + R + \frac{R^2}{\phi^2}\right). \tag{19}$$

This time, we find

$$R^2 = \phi, \tag{20}$$

so R is the square root of the Golden Ratio. The third column of table 4 lists the fractional curve lengths according to the "mirror symmetry" proposal. We'll explore yumi shapes whose curve lengths are consistent with the "mirror symmetry" proportions. Aside from tradition, there is pragmatic sense as well: Yumishi are quite aware that the stability of a yumi against out of plane flipping is precarious when the top curve is too long and too deep.

What are the proportions of curve elevations? The "consensus" elevations in (8) are a starting point. We see that the top concave curve is shallower than the convex curve below it by a factor of $\frac{3.2}{3.9} \approx .82$, and the bottom concave curve, shallower than the convex curve above it by the factor $\frac{1.0}{1.9} \approx .53$. This is qualitatively similar to the "mirror symmetry" proportions of curve lengths. Under mirror symmetry proportions, the top curve is shorter than the curve below it by the factor of $\frac{1}{\sqrt{\phi}} \approx .79$. The bottom curve is shorter than the curve above by the same factor. For a first trial, we propose these same proportions apply to respective curve *elevations* as well. Next, look at the proportions between the two convex curves. For the "consensus" of Symanski yumi, the upper convex curve is deeper than the lower by the factor $\frac{3.9}{1.9} \approx 2.1$. Under "mirror symmetry" proportions, the upper convex curve is longer than the longer by the factor $\phi \approx 1.61803$ (Golden Ratio). For our proposed shape, we impose the Golden ratio on the proportions of the respective curve elevations as well. This leaves the very shallow middle curve. The proposed sequence of curve elevations takes the form

$$-\frac{H}{\sqrt{\phi}},\ H,\ -h,\ \frac{H}{\phi},\ -\frac{H}{\phi^{\frac{3}{2}}}, \tag{21}$$

or inserting the explicit numerical approximation to the Golden Ratio,

$$-.786H,\ H,\ -h,\ .618H,\ -.486H. \tag{22}$$

Here, H is the elevation of the upper convex curve, and h, the depth of the middle cure. Both are determined so that two conditions hold: (i) The height of the yumi shape at the grip relative to the "plumb line" between the tips is $14.2cm$. (ii) The shape never descends below the "plumb line" between tips. If the shape is *tangent* to the plumb line at the upper tip, the determination of shape is *unique*. This shape is depicted in the first panel of figure 8. The photograph below the computer plots depicts a Shibata XX yumi after re-conditioning by Don Symanski. The photograph was offered to me as an example of a well proportioned yumi.

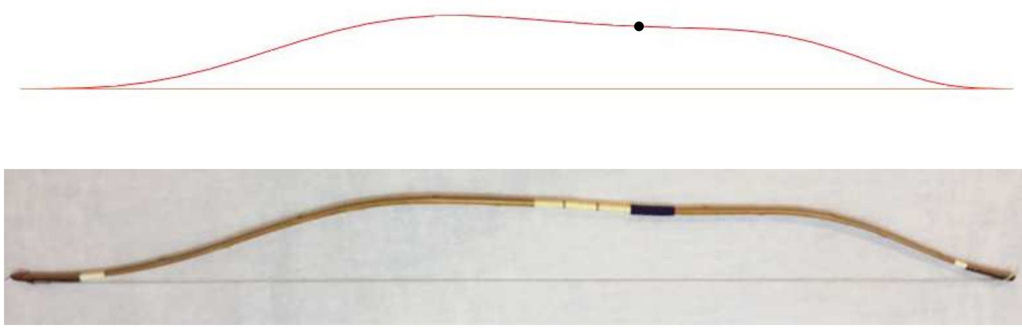

Figure 8: Yumi shape with "mirror symmetry" proportions. The "plumb line" between the tips of the shape does *not* represent the line of the tsuru. The tsuru would be about 2 cm below it.

III. Zanshin

Phidia's original division of a line segment according to the Golden ratio is a cairn marking the head the trail we walk. We determine a yumi shape whose curve proportions are uniquely defined in relation to each other. The essence of the aesthetics is a "continuity of proportions." In the words of Hashimoto sensei:

"Forms arise and give rise to other forms. Proportions are preserved so there is a continuity between the generations. Perhaps the ancient yumishi by freely following their intuitions and historical experiences were transparent in their hearts to the Nature within them and all around them. Perhaps the signature of Nature written on their hearts is imprinted in their yumi (bow)."

As mentioned before, the continuity of proportions is a class of *symmetries*. In a broader sense, *symmetries refer to aspects of phenomena which remain the same under a change of perspective*. Is this why we so readily respond to to them, our hearts recognizing them before our minds catch up? I've been exposed to meditation and insight traditions by the teacher Ryushin sensei, pointing to "radical impermanence." What is that? It is not "things change." In the words of another meditation teacher, Culadasa: "There are no 'things', only process." Ryushin sensei said: "The opposite of duality is not unity, but infinity." And yet, the sense that there is some sort of "still point" in all the diversity and movement: Here is a poetic but true fact from mathematics: Think of an ocean covering the whole world and there is always some shifting pattern of currents. At all times there is always somewhere a place with no movement. Symmetries as "still points"?

Along the way, practicalities insinuated themselves: The relationship of yumi design to segmented bamboo, nodes as convenient markers of inflection points along the unbraced yumi during the yumishi's inital layup process, and in the braced yumi, the shortness of the top concave curve relative to the big convex curve below it. Do these practicalities displace an aesthetic sense "emergent from a ground of feeling prior to words and forms"? Perhaps it does not *have* to be one or the other. Perhaps "practicality" and "heart" converge to the same place and that will *never* be explained. Nor need it be.