Precalculus CP: Summer Packet

Due on the first day of class

Name: ________________________________

Welcome to Precalculus!

Since so much of this course relies on the concepts you have learned in Algebra 1, Geometry, and Algebra 2, this packet has been designed to help you brush up on your skills so you can hit the ground running in August. For each section, I have provided an overview of the topic and a couple model problems to try to help you. Because of all the overview, this assignment is not as long as it looks! You are also welcome to use any online resource at your disposal (I especially recommend Khan Academy). If you choose to use an online resource, please indicate on which problems you used it, so I know where you had some difficulty. It's okay to get help– in fact, I expect you to in some cases– just be honest about when and where you did.

As of right now, I'm supposed to teach this class in the fall, and **I plan to count this assignment as a series of homework grades. There will also be a quiz on this content within the first few days of class.** However, things often change over the summer; no matter who your teacher is in the fall, assume that this assignment will count as a grade and that the content will be assessed in some way. I'm using this packet both for you to practice your math skills and for me to figure out any problem areas I should address right away. This material will be used extensively throughout the year, so it's crucial that it stays fresh in your mind.

You must show all work to receive full credit, and work should be completed neatly and thoroughly, preferably in pencil. In the interest of saving some paper, I didn't provide a lot of room to complete these problems, so **please work on separate sheets of paper, and attach them to this packet before submitting.** You shouldn't need to use your calculator at all for most of these problems.

If you have any questions at all over the summer, please reach out to me! My email is **rcox@theproutschool.org**. I would be happy to Zoom with you on a case-by-case basis if you're having difficulty.

Have a great summer!

- Mr. Cox

Some helpful resources:

- khanacademy.org for content review
- printablepaper.net for graph paper
- Me if you need clarification on the directions or have any other questions

1 Expressions

1.1 Simplifying Radical Expressions

We'll deal with the simplest type of radical: *square roots*. There are two important properties to remember:

$$1.\ \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \qquad 2.\ \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

When we simplify square roots, we should separate the radicand (the stuff underneath the square root) into factors, one of which should be a perfect square. This will let us use either one of the properties above to simplify our expression. For example:

$$\begin{aligned} \sqrt{8} &= \sqrt{4 \cdot 2} \\ &= \sqrt{4} \cdot \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

Another property of radicals is the ability to *rationalize* either the numerator or denominator of a rational expression, wherever a radical expression occurs[1]. It's not "illegal" to have a radical in the denominator, but for now, it's not very useful. For a single radical term, we multiply both halves of the fraction by that same radical. For example:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

If the part of the rational expression containing a radical has two terms, we have to multiply by the *conjugate* of that term. "Conjugate" is just a fancy word that means **"keep the same terms and switch the sign in between."** So:

$$\frac{1}{3+\sqrt{3}} = \frac{1}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{3-\sqrt{3}}{6}$$

This will always create a difference of squares in the numerator. If you remember the shortcut right now, great! If not, FOIL as usual and we'll cover it later.

1-6. Simplify the following expressions. Rationalize the denominator when necessary.

1. $\sqrt{48}$

2. $\sqrt{\frac{9}{2}}$

$\sqrt{100y^4}$

4. $\frac{2}{\sqrt{3}}$

5. $\frac{2}{\sqrt{2}+3}$

6. $\frac{1}{\sqrt{x}+1}$

[1] You won't be rationalizing the numerator now- that will be a useful strategy when you get to calculus.

1.2 Simplifying Rational Expressions

We have been dealing with rational expressions since elementary school...we've just been calling them fractions until now! The term "rational expression" is a more general term, since "fraction" often refers to expressions with numbers (e.g. $\frac{1}{2}$, $\frac{7}{3}$). When we add, subtract, multiply, or divide rational expressions, we work with them as if they were just more complicated fractions.

When adding or subtracting rational expressions, we must find a common denominator first, before adding across. Like with fractions, terms must have like denominators before we add or subtract.

$$\begin{aligned}
\frac{2}{x+1} + \frac{3}{x} &= \frac{2}{x+1} \cdot \frac{x}{x} + \frac{3}{x} \cdot \frac{x+1}{x+1} && \text{Make a common denominator} \\
&= \frac{2x}{x(x+1)} + \frac{3(x+1)}{x(x+1)} \\
&= \frac{2x+3x+3}{x(x+1)} && \text{Distribute and add} \\
&= \frac{5x+3}{x^2+x} && \text{Simplify}
\end{aligned}$$

Recall that division of rational expressions is multiplication by the divisor's reciprocal. You may have heard the expression "keep, change, flip" from past teachers. With rational expressions, however, we may see it written as a *complex fraction.* The most helpful thing to do with a complex fraction is rewrite as "normal" division.

$$\begin{aligned}
\frac{\frac{x}{x+5}}{\frac{3}{x-2}} &= \frac{x}{x+5} \div \frac{3}{x-2} && \text{Rewrite as division} \\
&= \frac{x}{x+5} \cdot \frac{x-2}{3} && \text{Reciprocal of the second term, multiply} \\
&= \frac{x \cdot (x-2)}{(x+5) \cdot 3} \\
&= \frac{x^2-2x}{3x+15} && \text{Simplify}
\end{aligned}$$

7-12. Simplify the following rational expressions. All answers should be reduced fully. Hint: In some problems, it will be easier to factor and simplify *before* completing the problem.

7. $\frac{2}{3} + \frac{1}{5}$

8. $\frac{3}{1-x} + \frac{5}{1+x}$

9. $\frac{x}{x+5} - \frac{2}{x-3}$

10. $\frac{2}{3} \cdot \frac{6}{5}$

11. $\frac{x^2+2x-3}{x+2} \cdot \frac{x^2+2x}{x^2-1}$

12. $\frac{\frac{(x+2)^2}{6x^2}}{\frac{3x}{x^2-4}}$

1.3 Simplifying Exponents

There are a few rules involving exponents that you've learned in the past:

Name	**Operation**	**General**	**Example**
Product of powers	Add exponents	$x^a \cdot x^b = x^{a+b}$	$x^2 \cdot x^5 = x^7$
Quotient of powers	Subtract exponents	$\frac{x^a}{x^b} = x^{a-b}$	$\frac{x^5}{x^2} = x^3$
Power to a power	Multiply exponents	$(x^a)^b = x^{ab}$	$(x^2)^3 = x^6$
Negative exponent	Reciprocals	$x^{-a} = \frac{1}{x^a}$	$2^{-1} = \frac{1}{2}$
Rational exponents	Roots	$x^{a/b} = \sqrt[b]{x^a}$	$x^{1/2} = \sqrt{x}$; $x^{2/3} = \sqrt[3]{x^2}$
Zero power		$x^0 = 1$	
Identity		$x^1 = x$	

When we simplify expressions involving exponents, we simply apply these rules. For example:

$$\frac{(a^2b)^2}{ab^3} = \frac{a^4b^2}{ab^3} = \frac{a^3}{b}$$

13-18. Simplify the following expressions, leaving only positive exponents in your final answer.

13. $a^4 \cdot a^3$

14. $(3x^{-1}y^2z^{-2})^3$

15. $\frac{3x^2}{18x}$

16. $(a^{1/2}b^{-}2)^3$

17. $t^{-1/2}$

18. $\frac{(4x^2)^{1/2}}{2x}$

2 Polynomials

2.1 Factoring out a Common Term

This is the simplest form of factoring. We want to remove anything common to all terms of the polynomial. In a way, we're dealing with the Distributive Property in reverse: $ab + ac = a(b + c)$. For example,

$$4x^2 + 16x = 4x(x + 4)$$

because our greatest common factor is $4x$. We're *dividing out* the greatest common factor in order to remove it from the polynomial.

19-20. Factor out the greatest common factor from the following polynomials.

19. $9xy^2 - 27x^2y$

20. $x^3 + 2x^2 + 5x$

2.2 Factoring a trinomial into two binomials $(ax^2 + bx + c)$

You may have learned this method through guess and check, but you should never have to guess in math! We're going to work this through what is called *master product factoring.* With this method, **we need factors of a · c that add up to b.** No matter the leading coefficient, we can always *factor by grouping.* Consider the trinomial $x^2 + 3x - 4$.

Our potential factor pairs are $\{1, 4\}$ or $\{2, 2\}$. We choose 1 and 4 because when multiplied, they equal 4, but when added, $4 + (-1) = 3$. **Tip:** When the constant term c is negative, the factors must have opposite signs. The sign of the linear term bx is the sign of the bigger factor. Next, we'll expand our linear term bx using the factor pair we chose, and continue from there:

$$\begin{aligned} x^2 + 3x - 4 &= x^2 + 4x - x - 4 && \text{Expand linear term as the sum of the two factors} \\ &= (x^2 + 4x) + (-x - 4) && \text{Pair off terms: grouping} \\ &= x(x + 4) - 1(x + 4) && \text{Factor the GCF out of each pair} \\ &= (x - 1)(x + 4) && \text{Factor out common term} \end{aligned}$$

Let's consider another example:

$$\begin{aligned} 2x^2 + 7x - 4 &= 2x^2 + 8x - x - 4 && \text{Factors pairs of } 2 \cdot 4 = 8 : \ (1, 8) \text{ or } (2, 4) \\ & && \text{Expand linear term as the sum of the two factors} \\ &= (2x^2 + 8x) + (-x - 4) && \text{Pair off terms: grouping} \\ &= 2x(x + 4) - 1(x + 4) && \text{Factor the GCF out of each pair} \\ &= (2x - 1)(x + 4) && \text{Factor out common term} \end{aligned}$$

Notice that the factors in parentheses in the second-to-last step MUST match. You might have also noticed that, in the first example, our final factored answer matches the factor pair that we chose. If you're comfortable making that jump, you can use that shortcut only when the leading coefficient $a = 1$. For example, $x^2 + 4x - 5 = (x + 5)(x - 1)$.

15-18. Factor the following polynomials into two binomial factors.

15. $3x^2 - x - 2$
16. $y^2 + y - 12$
17. $x^2 - 8x + 7$
18. $5t^2 + 19t + 12$

2.3 Factoring special cases

There are three special cases you should know well:

Name	**Form**	**Factored form**	**Example**	
Difference of Squares	$a^2 - b^2$	$= (a+b)(a-b)$	$x^2 - 16$	$= (x+4)(x-4)$
Sum, squared	$a^2 + 2ab + b^2$	$= (a+b)^2$	$x^2 + 6x + 9$	$= (x+3)^2$
Difference, squared	$a^2 - 2ab + b^2$	$= (a-b)^2$	$x^2 - 6x + 9$	$= (x-3)^2$

Notice how there's no "sum of squares"? We learned that $a^2 + b^2$ factors into terms with *imaginary* solutions. In this section we're concerned only with real number solutions, so we won't worry about imaginary solutions right now. Later in Precalculus you will review complex numbers.

21-26. Factor the following special cases of polynomials.

21. $x^2 - 4$
22. $x^2 - 25$
23. $t^2 - 10t + 25$
24. $t^2 + 8t + 16$
25. $y^2 + 2t + 1$
26. $y^2 - 4y + 4$

2.4 Solving Polynomial Equations

When we solve equations, we usually try to isolate the variable. With polynomial equations, we frequently have more than one variable in the expression, and so isolation is much more difficult at first. That is, until we factor the expression. There is one important property that will help us:

$$\text{If } a \cdot b = 0, \text{ then either } a = 0 \text{ or } b = 0$$

This property ONLY works when the equation is set equal to zero, so we should set the equation equal to zero first, before attempting to factor. It also works for any number of factors. For example:

$$\begin{aligned} x^3 + 4x^2 &= 5x \\ x^3 + 4x^2 - 5x &= 0 \quad \text{Set equal to zero} \\ x(x^2 + 4x - 5) &= 0 \quad \text{Factor out GCF} \\ x(x+5)(x-1) &= 0 \quad \text{Factor trinomial} \end{aligned}$$

From this, we can use our property.

$$\begin{array}{rcl|rcl|rcl} x &=& 0 & x+5 &=& 0 & x-1 &=& 0 \\ & & & x &=& -5 & x &=& 1 \end{array}$$

So, our solutions (also called *roots* or *zeroes*) are $x = 0, 1, -5$.

Of course, with a quadratic function ($ax^2 + bx + c$), we can also use the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This can be used any time, but if it's easier to factor, take the easy way out.

27-30. Solve the following equations. You may need the quadratic formula if factoring does not work.

27. $90x^4 = 10x^2$

28. $3x^2 + x = 1$

29. $x^3 - 7x^2 + 6x = 0$

30. $x^2 + 18x + 81 = 0$

3 Functions

3.1 Linear Equations

There are three forms of a linear equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y &= mx + b && \text{Slope-intercept form} \\ Ax + By &= C && \text{Standard form } (A, B, C \text{ must be integers}) \end{aligned}$$

If you start with two points, you'll be able to work your way algebraically through all three forms. For example, given the point $(0, 1)$ and $(2, -3)$, we can first calculate slope to be $m = \frac{-3-1}{2-0} = \frac{-4}{2} = -2$. Then, start with point-slope form:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 3 &= -2(x - 2) && \text{Substitute either point/Point-slope form} \\ y + 3 &= -2x + 4 \\ y &= -2x + 1 && \text{Slope-intercept form} \\ 2x + y &= 1 && \text{Standard form} \end{aligned}$$

As you go through the steps, notice that you arrive at each form of the line.

At times, you may also need to compare two lines. We can do that by comparing their slopes. Two lines are *parallel* if their slopes are equal ($m_1 = m_2$). Two lines are *perpendicular* if their slopes are opposite in sign and reciprocals ($m_1 = -\frac{1}{m_2}$).

31-33. Write the equations below in slope-intercept form. Then, graph the line, labeling the y-intercept and at least one other point.

31. $2x + 3y = 9$

31. $y = \frac{1}{2}x - 3$

32. $y - 2 = -\left(x + \frac{1}{3}\right)$

34-35. Find the equation of the line in a) point-slope, b) slope-intercept, and c) standard form which pass through the given points.

34. $(3, 1), (1, 3)$

35. $(2, -2), (0, -1)$

36-37. Are the lines defined by the points provided parallel or perpendicular to each other, or neither?

36. $\begin{cases} L_1 : (-1,-1),(2,-1) \\ L_2 : (3,3),(3,0) \end{cases}$

37. $\begin{cases} L_1 : (-2,-1),(1,4) \\ L_2 : (3,0),(0,5) \end{cases}$

3.2 Properties of Functions and Function Notation

We can think about a function as a machine with an operating system. A function f (which is the name of our machine) is a rule (operating system) relating inputs to outputs. Every input x in the *domain* must go to exactly one output y in the *range*. It's okay to have two inputs go to the same output–we can have two computer commands for the same function, for example–but we can never have one input go to two different outputs.

We typically notate our functions algebraically by the function's name and the variable it takes as an input: $f(x)$, for example. Then we define the rule: $f(x) = x^2 + 1$. So, this function's "operating system" takes every input, squares it, and adds one. It doesn't matter what we put inside. Even this is perfectly acceptable:

$$f(\ \ ) = (\ \ )^2 + 1$$

Which, I think, shows even more clearly that whatever I plug in to f as an input is what is applied to the function's rule:

$$f(2x) = (2x)^2 + 1 = 4x^2 + 1$$

$$f(3) = 3^2 + 1 = 10$$

We call this process *evaluating* a function. We can also talk about a function's *domain*. This is the set of all possible inputs (x-values) that can be plugged into the function. This can be done *explicitly* if the domain has a finite number of values, but we will focus more on the *implicit domain*; that is, given a function f, can we tell which values must be excluded from the domain? Fortunately, we can follow some rules:

- The domain of any **polynomial** function (ex: $f(x) = 3x^3 + 8x^2 - 7$) is all real numbers, written either in interval notation as $(-\infty, \infty)$ or in set notation as $\mathbb{R}$.
- The domain of any **rational** function (ex: $f(x) = \frac{x}{x+1}$) excludes any x that makes the denominator zero. If this number is a, this is written in interval notation as $(-\infty, a) \cup (a, \infty)$.
- The domain of any **radical** function with an *even index* (ex: $f(x) = \sqrt{x}$) is an inequality such that the radicand is greater than or equal to zero. For the example just given, $x \geq 0$, so we write this in interval notation as $[0, \infty)$.
- The domain of any **radical** function with an *odd index* (ex: $f(x) = \sqrt[3]{x}$) is all real numbers.

You should ask yourself two questions when finding the domain of any of these functions:

1. Do I have a rational expression anywhere in my function?
2. Do I have a square root anywhere in my function?

If you answered "no" to both questions, the domain is $\mathbb{R}$. Otherwise, follow the appropriate steps to find the domain.

38-41. Let $f(x) = x^2 + 2x - 1$. Find the following, and simplify completely:

38. $f(0)$
39. $f(-3)$
40. $f(2t-1)$
41. $f(2t)$

42-45. Let $g(x) = \dfrac{1}{2x-1}$. Find the following, and simplify completely:

42. $g(3)$
43. $g(-1)$
44. $g(2t-1)$
45. $g(t^2)$

46-48. Find the domain of the following functions and express your answers using interval notation.

46. $f(x) = \dfrac{1}{x^2-4}$
47. $g(x) = \sqrt{x+3}$
48. $h(t) = t^3 + 2t^2 - 4t + 1$

3.3 Inverse functions

We already know about inverse functions without explicitly calling them inverses: the square root is the inverse of squaring something, and adding 1 is the inverse of subtracting 1. What do these have in common? *These are functions that "undo" each other.* There are five steps to finding the inverse of a function $f(x)$:

1. Rewrite $f(x)$ as y.
2. Switch the places of x and y.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.
5. Check that your result is the inverse by showing that $f\big(f^{-1}(x)\big) = x$ and $f^{-1}\big(f(x)\big) = x$.

For example, let's find the inverses of $f(x) = 2x - 3$ and $g(x) = 2x^2 + 5$.

$$\begin{aligned} f(x) &= 2x - 3 \\ y &= 2x - 3 && \text{Rewrite } f(x) \text{ as } y \\ x &= 2y - 3 && \text{Switch } x,\ y \\ x + 3 &= 2y && \text{Solve for } y \\ \frac{x+3}{2} &= y \\ \frac{x+3}{2} &= f^{-1}(x) && \text{Replace } y \text{ with } f^{-1}(x) \end{aligned}$$

$$\begin{aligned} g(x) &= 2x^2 + 5 \\ y &= 2x^2 + 5 && \text{Rewrite } g(x) \text{ as } y \\ x &= 2y^2 + 5 && \text{Switch } x,\ y \\ x - 5 &= 2y^2 && \text{Solve for } y \\ \frac{x+5}{2} &= y^2 \\ \pm\sqrt{\frac{x+5}{2}} &= y \\ \sqrt{\frac{x+5}{2}} &= y && \text{Positive case only} \\ \sqrt{\frac{x+5}{2}} &= g^{-1}(x) && \text{Replace } y \text{ with } g^{-1}(x) \end{aligned}$$

Verifying that these functions are in fact inverses is left as extra practice for you.

49-50. Find the inverse of each function and verify that your result is the inverse. The inequality listed in 50 is not critical for your knowledge of the problem and does not affect the solution process. For now, it is a formality to ensure that the function really does have an inverse.

49. $f(x) = 3x + 4$

50. $h(x) = (5x - 2)^2,\ x \geq 2$