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Reliability Models Using the Composite Generalizers of Weibull Distribution

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Abstract

In this article, we study the composite generalizers of Weibull distribution using exponentiated, Kumaraswamy, transmuted and beta distributions. The composite generalizers are constructed using both forward and reverse order of each of these distributions. The usefulness and effectiveness of the composite generalizers and their order of composition is investigated by studying the reliability behavior of the resulting distributions. Two sets of real-world data are analyzed using the proposed generalized Weibull distributions.

Keywords Weibull distribution · Exponentiated distribution · Kumaraswamy distribution · Beta distribution · Transmutation map · Composition map

1 Introduction

A number of standard probability distributions have been found to be useful in analyzing data from various fields including engineering, medicine, economics and finance

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among others. However, generalizing these standard distributions using the genesis of some univariate distribution have produced more flexible distributions compared to the base line distributions. To this end, several attempts have been made by many authors to propose methods for generating new families of distributions. Readers are referred to Tahir and Nadarajah [55] and Tahir and Cordeiro [56] for more details. Special attention is paid in the extreme value distributions due to their popularity in the applied sciences. Extreme value distributions are of significant importance in studying the impact to the observed system. Needless to say, these types of probability structures are used to capture the behavior of rare events. Finance and insurance are major fields of research to observe the importance of extreme events but they are equally likely to occur in the field of structural engineering, materials science, and medical sciences [10]. The theory of extreme value originated in the last century as a tool to study the limiting distribution of maximum or minimum values.

Assume that $X_1, X_2, \ldots, X_n$ is a sequence of iid (independent and identically distributed) random variables with common cumulative distribution function (cdf), $F(x)$. Let

$$M_n = \max\{X_1, X_2, \cdots, X_n\}.$$

One is interested in the behavior of M_n as the sample size n increases to infinity.

$$\begin{aligned}\Pr\{M_n \le x\} &= \Pr\{X_1 \le x, X_2 \le x, \cdots, X_n \le x, \}\\ &= \Pr\{X_1 \le x\}\Pr\{X_2 \le x\}\cdots\Pr\{X_n \le x\}\\ &= [F(x)]^n.\end{aligned}$$

Suppose there are sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\Pr\left\{\frac{(M_n - b_n)}{a_n} \le x\right\} \to G(x) \text{ as } n \to \infty.$$

If $G(x)$ is a non-degenerate distribution function then it will belong to one of the following fundamental type of classic extreme value distribution:

– Type I (Gumbel distribution);
– Type II (Fréchet distribution);
– Type III (Weibull distribution).

Recent development has been focused on combining the genesis of more than one distributions to generalize a classical distribution which results into a more complex distribution with several parameters. In this study, we attempt to assess the effectiveness of the order of compositions of different generalizers taking Weibull distribution as our base model. In particular, we study possible differences in reliability with respect to the order of compositions. Due to the presence of incomplete beta function the composite generalizer involving beta generalizer followed by any other generalizer is mathematically challenging. In this report, we provide distributional compositions including the beta generalizer followed by three different generalizers to study their feasibility and applicability.

The Weibull distribution is a popular distribution named after Waladdi Weibull, a Swedish physicist. He applied this distribution in 1939 to analyze the breaking strength of materials. Since then, it has been widely used for analyzing lifetime data in reliability engineering and is one the most prominent distribution to model such data. A random variable X is said to have a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ if its cdf, $G(x)$, and probability density function (pdf), $g(x)$, are given by

$$G(x) = 1 - \exp\{-(\alpha x)^{\beta}\} \tag{1}$$

and

$$g(x) = \beta\alpha^{\beta} x^{\beta-1} \exp\{-(\alpha x)^{\beta}\}, \tag{2}$$

respectively. The Weibull distribution has an undeniable popularity in probability and statistics due to its versatile nature of modeling real world data. Yet there are many cases where the classical Weibull distribution is unable to capture the true phenomenon under study. Therefore, several of its generalizations have been proposed and studied. A generalized form of Weibull distribution is obtained by inducting one or more parameter(s) to the 2-parameter Weibull distribution. It has been proven that several of these generalized distributions are more flexible and are capable of modeling real world data better than the classical Weibull distribution. A state-of-the-art survey on the class of such generalized Weibull distributions can be found in Lai et al. [35] and Nadarajah [41].

This article is unfolded as below: In Sect. 2, we provide a review of commonly used generalizers. In Sect. 3, we study the generalizations of Weibull distribution using a single generalizer. Section 4 discusses the compounding of two generalizers using Weibull as a base distribution, and Sect. 5 explores the elements of reliability analysis. In Sect. 6, we assess the order of compositions via real world data analysis. Lastly, Sect. 7 provides some concluding remarks.

2 Review of Key Generalizers

In the last few decades, there has been an increased interest among statisticians in defining new generators of univariate distributions. It has been done by adding one or more shape parameter(s) to a baseline distribution to provide greater flexibility in modeling data in applied sciences. Some well-known generators include Marshal–Olkin generated family (MO-G) by Marshall and Olkin [37], the beta-G by Eugene et al. [23] and Jones [29], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [16], McDonald-G (Mc-G) by Alexander et al. [4], gamma-G by Zografos and Balakrishnan [60], exponentiated generalized-G by Cordeiro et al. [14], Weibull-G by Bourguignon et al. [9], Lomax-G by Cordeiro et al. [12] among others. Recent study has been focused on compounding two generalizers and apply it to develop a new distribution. Many of these attempts have been very successful and appear to have wider applicability. Some notable references include, Marshall–Olkin–Kumaraswamy-G (MOKw-G) family of distributions by Handique et al. [27], Beta Weibull-G family of distributions of Makubate et al. [36], exponentiated Weibull-H by Cordeiro et al. [11], Kumaraswamy generalized-G Poisson family by Ramos et al. [50], transmuted Kumaraswamy-G family of distributions by Khan et al. [34], among others.

In the context of life time distribution with cdf $G(x)$, commonly used generalizations are:

– Exponentiated-G distribution
– Beta-G distribution
– Kumaraswamy-G distribution
– Transmuted-G distribution

In Sects. 2.1 to 2.4 we provide brief descriptions of each of these methods.

2.1 Exponentiated-G Distribution

For a random variable with cdf $G(x)$, the exponentiated class of distribution has cdf and pdf given, respectively, by

$$F_{EG}(x) = [G(x)]^{\nu}, \tag{3}$$

$$f_{EG}(x) = \nu g(x)\,[G(x)]^{\nu-1}, \tag{4}$$

for $\nu > 0$. Mudholkar and Srivastava [38] first proposed the exponentiated Weibull distribution to analyze bathtub failure data. This paper has been most seminal in the development of generalized distributions. Gupta et al. [25] also proposed a generalization of the standard exponential distribution and coined it as exponentiated exponential (EE) distribution. Gupta and Kundu [26] provides some of its mathematical properties. Nadarajah and Kotz [42] proposed the exponentiated gamma, exponentiated Fréchet and exponentiated Gumbel distributions using slightly different cdf of the last two distributions. Cordeiro et al. [13] proposed a new class of distribution which extends the exponentiated type distributions.

2.2 Beta-G Distribution

The beta distribution has been used to generalize any parent cdf $G(x)$ as

$$F_{BG}(x) = I_{G(x)}(a,b) = \frac{B_{G(x)}(a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt, \tag{5}$$

$$f_{BG}(x) = \frac{g(x)}{B(a,b)}[G(x)]^{a-1}\big[1-G(x)\big]^{b-1}, \tag{6}$$

where $B_{G(x)}(a,b) = \int_0^{G(x)} w^{a-1}(1-w)^{b-1}dw$ is incomplete beta function, $a > 0$ and $b > 0$ are shape parameters whose role is to introduce skewness and to vary tail weight. Evidently, the exponentiated-G distribution is a special case of the beta-G distribution when $b = 1$. Although this generalization involves some complexity due to the presence of incomplete beta function it has attracted a number of researchers to generalize some well-known distributions. Some notable references include Eugene et al. [23], Nadarajah and Kotz [43], Nadarajah and Gupta [44], Famoye et al. [24], Cordeiro et al. [15], Achcar et al. [1], Hanook et al. [28]) and many others.

2.3 Kumaraswamy-G Distribution

Generalization using Kumaraswamy distribution (also known as minimax distribution) has been discussed by several authors. In this method, if $G(x)$ denotes the baseline cdf of a random variable then a generalized class of distributions has its cdf and pdf given by

$$F_{KG}(x) = 1 - \left[1 - G(x)^c\right]^d, \tag{7}$$

$$f_{KG}(x) = cdg(x)G(x)^{c-1}[1 - G(x)^c]^{d-1}, \tag{8}$$

where $c > 0$ and $d > 0$ are two shape parameters. The beta and Kumaraswamy distributions share similar properties. For example, the Kumaraswamy distribution is unimodal, uniantimodal, increasing, decreasing or constant depending on the values of its parameters. A more detailed description, background, genesis, and properties of Kumaraswamy distribution are outlined in Jones [29], where the author highlighted several advantages of the Kumaraswamy distribution over the beta distribution. In particular, its normalizing constant, explicit formulas for the distribution and quantile functions, and random number generator. There have been many contributions to the theory and applications of the Kumaraswamy distribution in the literature. See for example, Kumaraswamy Weibull by Cordeiro et al. [17], Kumaraswamy generalized gamma by de Pascoa et al. [18], Kumaraswamy geometric by Akinsete et al. [3]), and a host of many others.

2.4 Transmuted-G Distribution

A random variable X with cdf $G(X)$ is said to have transmuted distribution if its cdf and pdf are, respectively, given by

$$F_{TG}(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2, \tag{9}$$

$$f_{TG}(x) = g(x)[1 + \lambda - 2\lambda G(x)], \tag{10}$$

where $|\lambda| \leq 1$ is called the transmuted parameter. Details about the construction of transmuted generalizations are provided in Shaw and Buckley [52]. Following the work by Aryal and Tsokos [7] on the transmuted extreme value distribution, a number of transmuted family of distributions have been proposed and discussed by many authors in the literature. Tahir and Cordeiro [56] provided a comprehensive list of contributed works on transmuted distributions. There has been a continued interest with transmuted generalization procedure due to its simple and effective way to generalize a base distribution.

The Weibull distribution has been generalized using all of the afore mentioned generalizations. Exponentiated Weibull distribution by Pal et al. [47], Beta Weibull distribution by Famoye et al. [24], Kumaraswamy Weibull distribution by Cordeiro et al. [17] and transmuted Weibull by Aryal and Tsokos [6]. Each of these generalizations resulted with more versatility to the Weibull distribution which could be adopted to model real world data in several disciplines including the reliability analysis. Not only the Weibull distribution but some of its modifications including the additive Weibull distribution by Xie and Lai [57], extended Weibull distribution by Xie et al. [58], modified Weibull (MW) distribution by Lai et al. [35], flexible Weibull distribution by Bebbington et al. [8], inverse Weibull distribution by Keller and Kamath [31] among others have also been generalized. See for example, transmuted modified Weibull distribution by Khan and King [32], beta modified Weibull distribution by Silva et al. [54], Kumaraswamy inverse Weibull distribution by Shahbaz et al. [53], transmuted new generalized Weibull distribution by Khan et al. [33], to name a few. In this article, we purpose to combine two generalizers in order to develop a composite generalizer of Weibull distribution and study its effectiveness to model real world data. We provide a comprehensive list of possible composite generalizers using the genesis of exponentiated, beta, Kumaraswamy and transmuted models.

3 Generalization of Weibull Distribution

In this section we provide mathematical expressions of the generalized Weibull distribution using exponentiated, beta, Kumaraswamy and transmuted methods. Each of these generalizers resulted into a wider class which includes many classical distributions as a special case as described within each of the following subsections.

3.1 Exponentiated Weibull Distribution

Pal et al. [47] studied the generalization of the Weibull distribution to develop the exponentiated Weibull (EW) distribution. The cdf and pdf of the EW distribution are given, respectively, by

$$F_{EW}(x) = [1 - \exp\{-(\alpha x)^{\beta}\}]^{\nu}, \tag{11}$$

$$f_{EW}(x) = \nu\beta\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}[1 - \exp\{-(\alpha x)^{\beta}\}]^{\nu-1}. \tag{12}$$

Several classical probability distributions are embedded in the EW distribution. The particular case for $\beta = 1$ is the exponentiated exponential (EE) distribution. The particular case for $\nu = 1$ is the Weibull distribution. The particular case for $\beta = 2$ is the Burr type X distribution. The particular case for $\beta = 2$ and $\nu = 1$ is the Rayleigh distribution. A comprehensive review, mathematical properties and area of applications of the EW distribution is provided in Nadarajah and Kotz [40].

3.2 Kumaraswamy Weibull Distribution

Corderio et al. [17] studied the generalization of the Weibull distribution by using the Kumaraswamy generalizer. The cdf and pdf of the Kumaraswamy Weibull (KW) distribution are given by

$$F_{KW}(x) = 1 - \left\{1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d}, \tag{13}$$

$$f_{KW}(x) = cd\beta\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}[1 - \exp\{-(\alpha x)^{\beta}\}]^{c-1} \times \left\{1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d-1}. \tag{14}$$

The KW distribution houses a number of submodels as its special case. For $\beta = 1$, KW reduces to the Kumaraswamy exponential distribution, for $\beta = 2$, KW reduces to Kumaraswamy Rayleigh distribution, for $d = 1$, KW reduces to exponentiated Weibull distribution. Similarly, for $\beta = 2, d = 1$ the KW reduces to exponentiated Rayleigh distribution, for $\beta = d = 1$ the KW distribution reduces to exponentiated exponential distribution, for $c = d = 1$ the KW reduces to Weibull distribution, for $\beta = 2, c = d = 1$, the KW reduces to Rayleigh distribution and for $\beta = c = d = 1$ the KW reduces to exponential distribution. Since many classical distributions are embedded, KW distribution has been found to be widely applicable in many different area of applied Sciences. Readers are referred to Cordeiro et al. [17] for details.

3.3 Beta Weibull Distribution

Famoye et al. [24] studied the generalization of the Weibull distribution by using the beta generalizer to develop the beta Weibull (BW) distribution. The cdf and pdf of the BW distribution are given by

$$F_{BW}(x) = I_{1-\exp\{-(\alpha x)^{\beta}\}}(a, b), \tag{15}$$

$$f_{BW}(x) = \frac{\beta\alpha^{\beta}}{B(a, b)}x^{\beta-1}\exp\{-b(\alpha x)^{\beta}\}[1 - \exp\{-(\alpha x)^{\beta}\}]^{a-1} \tag{16}$$

The BW distribution is an extended model to analyze more complex data and generalizes some classical distributions in the literature. In particular, the BW distribution contains the exponentiated Weibull distribution as special cases when $b = 1$. The Weibull distribution is a special case for $a = b = 1$. When $a = 1$, BW reduces to a Weibull distribution with parameters $\alpha b^{1/\beta}$ and β. The beta exponential distribution is also a special case for $\beta = 1$. The BW reduces to exponential distribution for $a = b = \beta = 1$. Readers can find mathematical properties and some applications in Famoye et al. [24].

3.4 Transmuted Weibull Distribution

Aryal and Tsokos [6] studied the generalization of the Weibull distribution to develop the transmuted Weibull (TW) distribution. The cdf and pdf of the TW distribution are given, respectively, by

$$
\begin{aligned}
F_{TW}(x) &= [1 - \exp\{-(\alpha x)^{\beta}\}][1 + \lambda \exp\{-(\alpha x)^{\beta}\}] \\
&= 1 + (\lambda - 1) \exp\{-(\alpha x)^{\beta}\} - \lambda \exp\{-2(\alpha x)^{\beta}\}, \qquad (17)\\
f_{TW}(x) &= \beta\alpha^{\beta} x^{\beta-1} \exp\{-(\alpha x)^{\beta}\}[1 - \lambda + 2\lambda \exp\{-(\alpha x)^{\beta}\}]. \qquad (18)
\end{aligned}
$$

The TW distribution extends some well known distributions in the literature. For $\lambda = 0$, TW reduces to the Weibull distribution; for $\beta = 1$, TW reduces to the transmuted exponential distribution; for $\beta = 2$ TW reduces to transmuted Rayleigh distribution; for $\beta = 2$ and $\lambda = 0$, TW reduces to Rayleigh distribution and for $\lambda = 0$, $\beta = 1$, TW reduces to exponential distribution. The TW distribution appears to have wider applicability in many different areas due to its mathematical tractability and versatile shapes. For mathematical characteristics and applications of transmuted Weibull distribution readers are referred to Aryal and Tsokos [6].

4 Composite Generalizers

The Weibull distribution has been generalized using a single generalizer through the genesis of exponentiated, beta, Kumaraswamy and transmuted distributions by several authors as described in Sect. 3. Some attempts have been made to compound two of these generalizers to develop a new generalized Weibull distribution. Example includes recent work by Eissa [22]: exponentiated Kumaraswamy Weibull distribution, Ebraheim [21] and Adam et al. [2]: exponentiated transmuted Weibull distribution, Oseghale et al. [46]: Kumaraswamy transmuted Weibull distribution, Cordeiro et al. [13]: beta exponentiated Weibull distribution, Pal and Tiensuwan [48]: beta transmuted Weibull distribution. To the best of our knowledge there has been no attempt to add a generalizer to beta Weibull distribution. In this study, we provide a comprehensive list of the generalized Weibull distribution by compounding two of these generalizers and study the reliability behavior to perform the effectiveness and usefulness of compounding. A synopsis of twelve generalized Weibull distribution with composite generalizers is provided below. The exponentiated beta Weibull (EBW), Kumaraswamy beta Weibull (KBW) and transmuted beta Weibull (TBW) are introduced for the first time. Although these distributions have resulted with some complex mathematical structures but appear to be very flexible to model data sets with extreme observations. The pdf of all twelve distributions is displayed in Fig. 1 for selected values of the parameters.

4.1 Exponentiated-Kumaraswamy Weibull Distribution

Using (3) and (4) to the Kumaraswamy Weibull distribution [(13), (14)], the cdf and pdf of exponentiated-Kumaraswamy Weibull (EKW) distribution are given, respectively, by

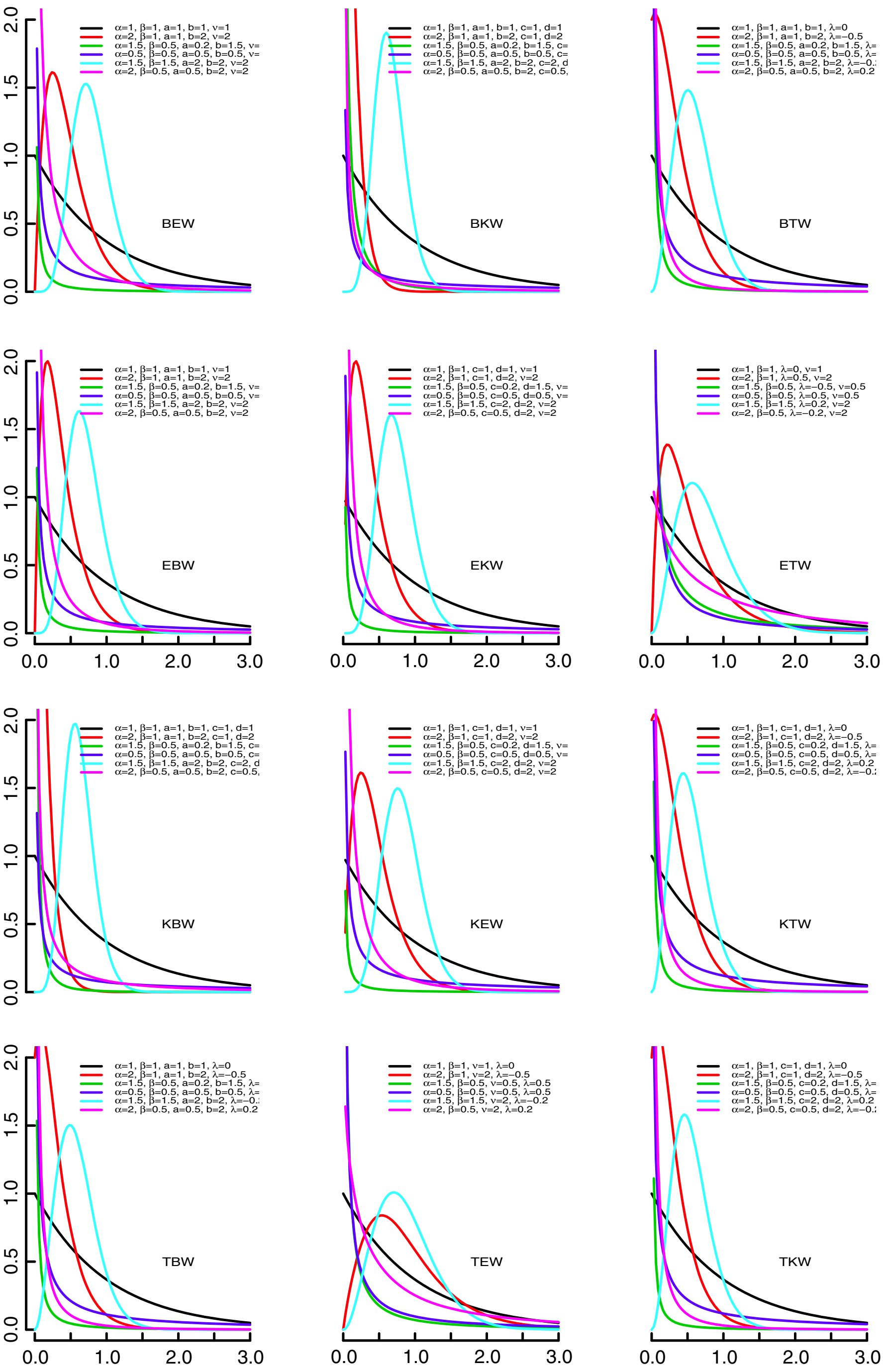

Fig. 1 PDF of the generalized Weibull distributions

$$F_{EKW}(x) = \left[1 - \left\{1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d}\right]^{\nu},$$
$$f_{EKW}(x) = cd\beta\nu\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}\left[1 - \left\{1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d}\right]^{\nu-1}$$
$$\times [1 - \exp\{-(\alpha x)^{\beta}\}]^{c-1}\left\{1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d-1}.$$

4.2 Exponentiated-Beta Weibull Distribution

Using (3) and (4) to the beta Weibull distribution [(15), (16)], the cdf and pdf of exponentiated-beta Weibull (EBW) distribution are given, respectively, by

$$F_{EBW}(x) = \left[I_{[1-\exp\{-(\alpha x)^{\beta}\}]}(a, b)\right]^{\nu},$$
$$f_{EBW}(x) = \frac{\beta\nu\alpha^{\beta}}{B(a, b)}x^{\beta-1}\exp\{-b(\alpha x)^{\beta}\}[1 - \exp\{-(\alpha x)^{\beta}\}]^{a-1}$$
$$\left[I_{[1-\exp\{-(\alpha x)^{\beta}\}]}(a, b)\right]^{\nu-1}.$$

4.3 Exponentiated-Transmuted Weibull Distribution

Using (3) and (4) to the transmuted Weibull distribution [(17), (18)], the cdf and pdf of exponentiated-transmuted Weibull (ETW) distribution are given, respectively, by

$$F_{ETW}(x) = \left[1 + (\lambda - 1)\exp\{-(\alpha x)^{\beta}\} - \lambda\exp\{-2(\alpha x)^{\beta}\}\right]^{\nu},$$
$$f_{ETW}(x) = \beta\nu\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}[1 - \lambda + 2\lambda\exp\{-(\alpha x)^{\beta}\}]$$
$$\times [1 + (\lambda - 1)\exp\{-(\alpha x)^{\beta}\} - \lambda\exp\{-2(\alpha x)^{\beta}\}]^{\nu-1}.$$

4.4 Kumaraswamy-Exponentiated Weibull Distribution

Using (7) and (8) to the exponentiated Weibull distribution [(11), (12)], the cdf and pdf of Kumaraswamy exponentiated Weibull (KEW) distribution are given, respectively, by

$$F_{KEW}(x) = 1 - \left[1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c\nu}\right]^{d},$$
$$f_{KEW}(x) = cd\nu\beta\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}\left[1 - \exp\{-(\alpha x)^{\beta}\}\right]^{c\nu-1}$$
$$\times \left[1 - [1 - \exp\{-(\alpha x)^{\beta}\}]^{c\nu}\right]^{d-1}.$$

4.5 Kumaraswamy-Beta Weibull Distribution

Using (7) and (8) to the beta Weibull distribution [(15), (16)], the cdf and pdf of Kumaraswamy beta Weibull (KBW) distribution are given, respectively, by

$$F_{KBW}(x) = 1 - \left[1 - \left\{I_{1-\exp\{-(\alpha x)^\beta\}}(a,b)\right\}^c\right]^d,$$

$$f_{KBW}(x) = \frac{cd\beta\alpha^\beta}{B(a,b)} x^{\beta-1}\exp\{-b(\alpha x)^\beta\}[1-\exp\{-(\alpha x)^\beta\}]^{a-1}$$
$$\times \left[I_{1-\exp\{-(\alpha x)^\beta\}}(a,b)\right]^{c-1}\left[1-\left\{I_{1-\exp\{-(\alpha x)^\beta\}}(a,b)\right\}^a\right]^{d-1}.$$

4.6 Kumaraswamy-Transmuted Weibull Distribution

Using (7) and (8) to the beta Weibull distribution[(17), (18)], the cdf and pdf of Kumaraswamy-transmuted Weibull (KTW) distribution are given, respectively, by

$$F_{KTW}(x) = 1 - \left[1 - \left\{1 + (\lambda-1)\exp\{-(\alpha x)^\beta\} - \lambda\exp\{-2(\alpha x)^\beta\}\right\}^c\right]^d,$$

$$f_{KTW}(x) = cd\beta\alpha^\beta x^{\beta-1}\exp\{-(\alpha x)^\beta\}[1-\lambda+2\lambda\exp\{-(\alpha x)^\beta\}]$$
$$\times \left[1+(\lambda-1)\exp\{-(\alpha x)^\beta\}-\lambda\exp\{-2(\alpha x)^\beta\}\right]^{c-1}$$
$$\times \left[1-\left\{1+(\lambda-1)\exp\{-(\alpha x)^\beta\}-\lambda\exp\{-2(\alpha x)^\beta\}\right\}^c\right]^{d-1}.$$

4.7 Beta-Exponentiated Weibull Distribution

Using (5) and (6) to the exponentiated Weibull distribution [(11), (12)], the cdf and pdf of beta-exponentiated Weibull (BEW) distribution are given, respectively, by

$$F_{BEW}(x) = I_{[1-\exp\{-(\alpha x)^\beta\}]^\nu}(a,b) = \frac{1}{B(a,b)}\int_0^{[1-\exp\{-(\alpha x)^\beta\}]^\nu} t^a(1-t)^{b-1}dt,$$

$$f_{BEW}(x) = \frac{\alpha\beta\nu}{B(a,b)}(\alpha x)^{\beta-1}\exp\{-(\alpha x)^\beta\}[1-\exp\{-(\alpha x)^\beta\}]^{a\nu-1}$$
$$\left[1-\left(1-\exp\{-(\alpha x)^\beta\}\right)^\nu\right]^{b-1}.$$

4.8 Beta-Kumaraswamy Weibull Distribution

Using (5) and (6) to the Kumaraswamy Weibull distribution [(13), (14)], the cdf and pdf of beta-Kumaraswamy Weibull (BKW) distribution are given, respectively, by

$$F_{BKW}(x) = I_{[1-\{1-[1-\exp\{-(\alpha x)^\beta\}]^c\}^d]}(a,b)$$
$$= \frac{1}{B(a,b)}\int_0^{1-\{1-[1-\exp\{-(\alpha x)^\beta\}]^c\}^d} t^a(1-t)^{b-1}dt,$$

$$f_{BKW}(x) = \frac{1}{B(a,b)}cd\beta\alpha^\beta x^{\beta-1}\exp\{-(\alpha x)^\beta\}[1-\exp\{-(\alpha x)^\beta\}]^{c-1}$$
$$\times \left\{1-[1-\exp\{-(\alpha x)^\beta\}]^c\right\}^{db-1}\left[1-\left\{1-[1-\exp\{-(\alpha x)^\beta\}]^c\right\}^d\right]^{a-1}.$$

4.9 Beta-Transmuted Weibull Distribution

Using (5) and (6) to the transmuted Weibull distribution [(17), (18)], the cdf and pdf of beta-transmuted Weibull (BTW) distribution are given, respectively, by

$$\begin{aligned} F_{BTW}(x) &= I_{[1+(\lambda-1)\exp\{-(\alpha x)^{\beta}\}-\lambda\exp\{-2(\alpha x)^{\beta}\}]}(a,b) \\ &= \frac{1}{B(a,b)}\int_0^{1+(\lambda-1)\exp\{-(\alpha x)^{\beta}\}-\lambda\exp\{-2(\alpha x)^{\beta}\}} t^{a}(1-t)^{b-1}dt, \\ f_{BTW}(x) &= \frac{1}{B(a,b)}\beta\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}[1+(\lambda-1)\exp\{-(\alpha x)^{\beta}\}-\lambda\exp\{-2(\alpha x)^{\beta}\}]^{a-1} \\ &\quad\times[1-\lambda+2\lambda\exp\{-(\alpha x)^{\beta}\}][\lambda\exp\{-2(\alpha x)^{\beta}\}-(\lambda-1)\exp\{-(\alpha x)^{\beta}\}]^{b-1}. \end{aligned}$$

4.10 Transmuted-Exponentiated Weibull Distribution

Using (9) and (10) to the exponentiated Weibull distribution [(11), (12)], the cdf and pdf of transmuted exponentiated Weibull (TEW) distribution are given, respectively, by

$$\begin{aligned} F_{TEW}(x) &= [1-\exp\{-(\alpha x)^{\beta}\}]^{\nu}\left[1+\lambda-\lambda[1-\exp\{-(\alpha x)^{\beta}\}]^{\nu}\right], \\ f_{TEW}(x) &= \nu\beta\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}[1-\exp\{-(\alpha x)^{\beta}\}]^{\nu-1} \\ &\quad\times\left[1+\lambda-2\lambda[1-\exp\{-(\alpha x)^{\beta}\}]^{\nu}\right]. \end{aligned}$$

4.11 Transmuted-Kumaraswamy Weibull Distribution

Using (9) and (10) to the Kumaraswamy Weibull distribution[(13), (14)], the cdf and pdf of transmuted-Kumaraswamy Weibull (TKW) distribution are given, respectively, by

$$\begin{aligned} F_{TKW}(x) &= \left[1-\left\{1-[1-\exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d}\right]\left[1+\lambda\left\{1-[1-\exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d}\right], \\ f_{TKW}(x) &= cd\beta\alpha^{\beta}x^{\beta-1}\exp\{-(\alpha x)^{\beta}\}[1-\exp\{-(\alpha x)^{\beta}\}]^{c-1}\left\{1-[1-\exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d-1} \\ &\quad\times\left[1-\lambda+2\lambda\left\{1-[1-\exp\{-(\alpha x)^{\beta}\}]^{c}\right\}^{d}\right]. \end{aligned}$$

4.12 Transmuted-Beta Weibull Distribution

Using (9) and (10) to the beta Weibull distribution [(15), (16)], the cdf and pdf of transmuted beta Weibull (TBW) distribution are given, respectively, by

$$\begin{aligned} F_{TBW}(x) &= I_{[1-\exp\{-(\alpha x)^{\beta}\}]}(a,b)\left[1+\lambda-\lambda I_{[1-\exp\{-(\alpha x)^{\beta}\}]}(a,b)\right], \\ f_{TBW}(x) &= \frac{1}{B(a,b)}\beta\alpha^{\beta}x^{\beta-1}\exp\{-b(\alpha x)^{\beta}\}[1-\exp\{-(\alpha x)^{\beta}\}]^{a-1} \\ &\quad\times\left[1+\lambda-2\lambda I_{[1-\exp\{-(\alpha x)^{\beta}\}]}(a,b)\right]. \end{aligned}$$

5 Reliability Analysis

The survival function, also known as the reliability function in engineering, of a probability distribution is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is the probability that the system will survive beyond a specified time. The reliability function $R(t)$ is defined by $R(t) = 1 - F(t)$, where $F(.)$ is the cdf of the distribution. Table 1 provides the expressions of the reliability function of all twelve generalized Weibull distribution.

In order to measure the effectiveness of the order of inducted generalizer we study the reliability behavior of their respective combinations. Table 2 shows reliability of different compositions of probability distributions for a given choice of parameters. In each compositions—Weibull (W) is a base distribution followed by exponentiated (E), transmuted (T), Kumarashwamy (K), and beta (B). Likewise, for every combinations, we present reliability for three sets of randomly selected parameters. Table 2 also depict a pairwise comparisons of compositions with same number of parameters with varying values. For example, BEW and EBW has been evaluated using same parameter values with three different combinations which are named BEW1, EBW1; BEW2, EBW2; and BEW3, EBW3. A pairwise comparison between BEW and EBW shows EBW with consistently higher reliability. Among all the comparison we made for this report, BKW and KBW show largest difference between the reliabilities with an average of 10% higher reliability of BKW composition with

Table 1 Expressions of reliability functions for twelve generalized Weibull distributions

Model	Reliability function
EKW	$1-\left[1-\left\{1-[1-\exp\{-(\alpha t)^{\beta}\}]^{c}\right\}^{d}\right]^{\nu}$
EBW	$1-\left[I_{[1-\exp\{-(\alpha t)^{\beta}\}]}(a,b)\right]^{\nu}$
ETW	$1-\left[1+(\lambda-1)\exp\{-(\alpha t)^{\beta}\}-\lambda\exp\{-2(\alpha t)^{\beta}\}\right]^{\nu}$
KEW	$\left[1-[1-\exp\{-(\alpha t)^{\beta}\}]^{c\nu}\right]^{d}$
KBW	$\left[1-\left\{I_{1-\exp\{-(\alpha t)^{\beta}\}}(a,b)\right\}^{c}\right]^{d}$
KTW	$\left[1-\left\{1+(\lambda-1)\exp\{-(\alpha t)^{\beta}\}-\lambda\exp\{-2(\alpha t)^{\beta}\}\right\}^{c}\right]^{d}$
BEW	$1-I_{[1-\exp\{-(\alpha t)^{\beta}\}]^{\nu}}(a,b)$
BKW	$1-I_{\left[1-\left\{1-[1-\exp\{-(\alpha t)^{\beta}\}]^{c}\right\}^{d}\right]}(a,b)$
BTW	$1-I_{[1+(\lambda-1)\exp\{-(\alpha t)^{\beta}\}-\lambda\exp\{-2(\alpha t)^{\beta}\}]}(a,b)$
TEW	$1-[1-\exp\{-(\alpha t)^{\beta}\}]^{\nu}\left[1+\lambda-\lambda[1-\exp\{-(\alpha t)^{\beta}\}]^{\nu}\right]$
TKW	$1-\left[1-\left\{1-[1-\exp\{-(\alpha t)^{\beta}\}]^{c}\right\}^{d}\right]\left[1+\lambda\left\{1-[1-\exp\{-(\alpha t)^{\beta}\}]^{c}\right\}^{d}\right]$
TBW	$1-I_{[1-\exp\{-(\alpha t)^{\beta}\}]}(a,b)\left[1+\lambda-\lambda I_{[1-\exp\{-(\alpha t)^{\beta}\}]}(a,b)\right]$

Table 2 Reliability values of the twelve different generalized Weibull distributions

Parameters	Distribution	*t*						
		0.000	0.500	1.000	1.500	2.000	2.500	3.000
$\alpha = 1, \beta = 0.37, \nu = 0.5, a = 1.5, b = 1.2$	BEW1	1.000	0.298	0.222	0.181	0.154	0.134	0.118
	EBW1	1.000	0.321	0.242	0.198	0.169	0.147	0.131
$\alpha = 1, \beta = 0.7, \nu = 0.5, a = 1.5, b = 1.6$	BEW2	1.000	0.253	0.128	0.073	0.045	0.028	0.019
	EBW2	1.000	0.312	0.168	0.100	0.062	0.040	0.027
$\alpha = 1, \beta = 0.9, \nu = 0.5, a = 1.5, b = 1.8$	BEW3	1.000	0.247	0.097	0.042	0.019	0.009	0.004
	EBW3	1.000	0.322	0.140	0.064	0.030	0.014	0.007
$\alpha = 0.4, \beta = 0.8, a = 1.5, b = 1.5, c = 1, d = 1$	BKW1	1.000	0.814	0.649	0.518	0.415	0.334	0.269
	KBW1	1.000	0.661	0.486	0.369	0.285	0.223	0.176
$\alpha = 0.7, \beta = 0.8, a = 1.6, b = 1.9, c = 1.8, d = 1.3$	BKW2	1.000	0.846	0.580	0.365	0.221	0.131	0.078
	KBW2	1.000	0.796	0.502	0.295	0.169	0.096	0.055
$\alpha = 0.9, \beta = 0.9, a = 1.2, b = 1.9, c = 1.8, d = 1.7$	BKW3	1.000	0.603	0.241	0.084	0.028	0.009	0.003
	KBW3	1.000	0.515	0.175	0.054	0.017	0.005	0.002
$\alpha = 0.3, \beta = 1, \lambda = -0.9, a = 0.7, b = 0.8$	BTW1	1.000	0.926	0.848	0.770	0.696	0.626	0.562
	TBW1	1.000	0.938	0.867	0.794	0.723	0.656	0.593
$\alpha = 0.3, \beta = 0.7, \lambda = 0.84, a = 1.6, b = 1.9$	BTW2	1.000	0.592	0.373	0.246	0.167	0.117	0.083
	TBW2	1.000	0.659	0.444	0.308	0.219	0.159	0.117
$\alpha = 1.2, \beta = 1.9, \lambda = 0.9, a = 2.7, b = 3.2$	BTW3	1.000	0.395	0.002	0.000	0.000	0.000	0.000
	TBW3	1.000	0.578	0.010	0.000	0.000	0.000	0.000

Table 2 continued

Parameters	Distribution	t						
		0.000	0.500	1.000	1.500	2.000	2.500	3.000
$\alpha = 0.2, \beta = 1.7, \nu = 0.5, c = 0.4, d = 1.9$	EKW1	1.000	0.402	0.270	0.190	0.136	0.098	0.070
	KEW1	1.000	0.314	0.197	0.132	0.091	0.063	0.044
$\alpha = 0.4, \beta = 1.9, \nu = 0.7, c = 0.6, d = 1.7$	EKW2	1.000	0.618	0.385	0.227	0.124	0.063	0.029
	KEW2	1.000	0.580	0.346	0.197	0.105	0.052	0.024
$\alpha = 0.6, \beta = 2.1, \nu = 0.4, c = 0.8, d = 1.9$	EKW3	1.000	0.445	0.193	0.067	0.017	0.003	0.000
	KEW3	1.000	0.333	0.120	0.036	0.008	0.001	0.000
$\alpha = 0.67, \beta = 1.2, \lambda = 0.96, c = 0.5, c = 1.6$	TKW1	1.000	0.128	0.032	0.009	0.003	0.001	0.000
	KTW1	1.000	0.195	0.055	0.016	0.004	0.001	0.000
$\alpha = 0.6, \beta = 1.8, \lambda = -0.9, c = 1.6, c = 1.6$	TKW2	1.000	0.994	0.915	0.665	0.347	0.132	0.039
	KTW2	1.000	0.997	0.939	0.729	0.418	0.173	0.054
$\alpha = 0.8, \beta = 1.3, \lambda = 0.6, c = 1.8, c = 1.2$	TKW3	1.000	0.836	0.495	0.240	0.107	0.047	0.020
	KTW3	1.000	0.795	0.441	0.207	0.092	0.040	0.017
$\alpha = 1.6, \beta = 1.2, \nu = 0.3, \lambda = 1$	ETW1	1.000	0.071	0.009	0.001	0.000	0.000	0.000
	TEW1	1.000	0.029	0.003	0.000	0.000	0.000	0.000
$\alpha = 0.6, \beta = 0.9, \nu = 0.4, \lambda = 1$	ETW2	1.000	0.247	0.125	0.068	0.039	0.023	0.014
	TEW2	1.000	0.154	0.069	0.035	0.019	0.011	0.006
$\alpha = 1.2, \beta = 1.7, \nu = 3.1, \lambda = 0.8$	ETW3	1.000	0.866	0.287	0.051	0.008	0.001	0.000
	TEW3	1.000	0.936	0.408	0.067	0.008	0.001	0.000

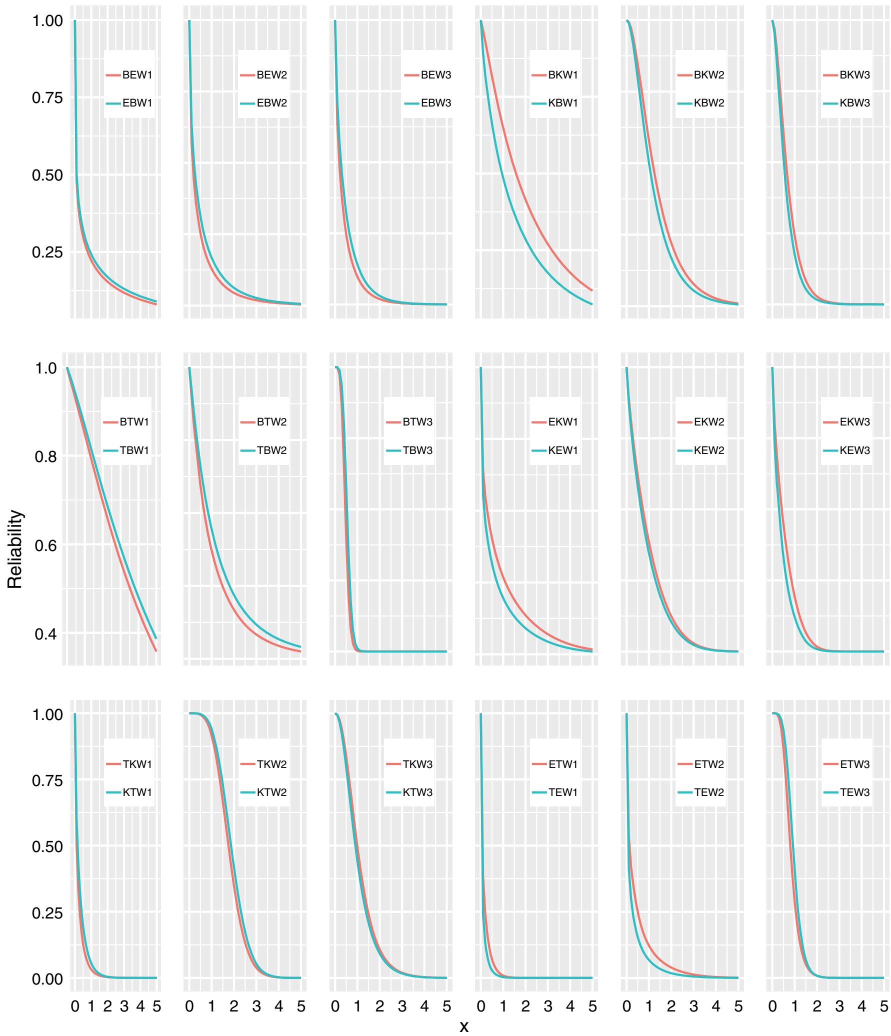

Fig. 2 Reliability plot for pairwise comparisons of composite generalizers

$\alpha = 0.4$, $\beta = 0.8$, $a = 1.5$, $b = 1.5$, $c = 1$, and $d = 1$. In addition, TBW, EKW, and TEW showed higher reliability in comparison with BTW, KEW, and ETW respectively. However, no obvious consistency was observed while comparing TKW and KTW for different choices of parameters.

In Table 2, first column displays a choice of parameter values, second column shows a pairwise selection of different compositions and third column provides reliability of respective distributions based on the choice of parameters.

In Fig. 2 we display the graphical presentation of the composite generalizers discussed in Table 2. Observe that in all three sets of parameters the order of the composition yield different reliability values.

6 Applications

The composite generalizers are effective way to develop a new probability distribution. It should be noted that the additional parameter(s) not always produce significantly better model. It is equally important to consider the order of composition to be selected to construct a composite generalizer (eg. BEW versus EBW). In order to compare the distributions, we consider the goodness-of-fit statistics namely the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson–Darling (A^*) and Cramér–Von Mises (W^*), Kolmogorov–Smirnov test statistics (D). These statistics are given by

$$\begin{aligned}
&AIC = -2\widehat{\ell} + 2p,\ BIC = -2\widehat{\ell} + p\log(n),\\
&HQIC = -2\widehat{\ell} + 2p\log\left[\log(n)\right],\ CAIC = -2\widehat{\ell} + 2pn/(n-p-1),\\
&A^* = \left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right]\right\},\\
&W^* = \left(\frac{1}{2n} + 1\right)\left[\sum_{j=1}^{n}\left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right],
\end{aligned}$$

and

$$D = \max_{1\le i\le n}\left|F(x_i) - \frac{i}{n}\right|$$

respectively, where $z_i = F(x_{(i)})$, p is the number of parameters, n is the sample size and the values $x_{(i)}$'s are the *ordered observations*. The smaller these statistics are, the better the fit is. Upper tail percentiles of the asymptotic distributions of these goodness-of-fit statistics are tabulated in Nichols and Padgett [45]. In this Section, we present two examples to illustrate the usefulness of the afore mentioned generalized Weibull distributions. All required computations are carried out using statistical program R 3.5 [51].

6.1 Example 1: Kevlar Data Modeling

Kevlar is a kind of plastic fiber from aromatic polyamide family. The molecules of Kevlar are ring like structure similar to benzene and are connected together based on a modified benzene-like ring structure. This fiber was invented in DuPont scientific laboratory in 1965 [20]. Based on the literature review, the interaction of Kevlar 49 fiber with sunlight will lead to degradation. Thus, it must be shielded from these radiations. According to Penn [49] Kevlar 49 fiber has better thermal stability for an organic material and the fiber can be used as a multifilament yarn. Kevlar fiber is increasingly used in aerospace, automobile, and defense industries due to good impact resistance and high damage tolerance as stated in Yeung and Rao [59]. In addition, the impact

Table 3 Statistical Summary of the Kevlar data

n	Minimum	Q_1	Median	Mean	Q_3	Maximum
76	0.0251	0.9048	1.7362	1.9592	2.2959	9.0960

Table 4 Estimated parameters and their standard errors (in parenthesis) for the Kevlar data

Model	α	β	a	b	c	d	ν	λ
TKW	0.444	0.824	–	–	1.585	2.392	–	− 0.719
	(1.007)	(0.532)	–	–	(1.433)	(6.905)	–	(0.263)
EKW	0.412	1.001	–	–	1.657	1.884	0.954	–
	(0.766)	(0.613)	–	–	(2.874)	(5.331)	(1.350)	–
BKW	0.608	1.095	1.115	0.864	1.299	1.178	–	–
	(1.408)	(0.333)	(0.182)	(9.731)	(0.872)	(11.005)	–	–
KW	0.725	1.156	–	–	1.348	0.739		
	(1.216)	(0.382)	–	–	(0.676)	(1.865)		
W	0.469	1.326	–	–	–	–	–	–
	(0.043)	(0.114)	–	–	–	–	–	–

properties can be increased by designing a hybrid material of carbon fibre and Kevlar 49 fibre. It is probably best known for its use in bulletproof vests, knifeproof and different kinds of anti ballistic materials, car tires, in strings of archary bows, boat and aircraft bodies. It's a material often noted as being "five times stronger than steel" on equal weight basis, Dorey [19]. Unlike most plastics it does not melt and decomposes only about at 450 °C. However, low temperature have no effect on Kevlar down to − 320 °C. Kevlar possesses a strong resistant against attacks from various chemicals though long term exposure to acid and bases will cause degrade it over time. The purpose of this study is to identify an appropriate probability distribution which could be used to model the reliability behavior of the Kevlar 49 fibre. We consider a data set of the life (in hours) of fatigue fracture of Kevlar 49/Epoxy strands at 373.9 kilopound per square inch (ksi) at 110 degree Celsius temperature that are subject to sustained pressure at 90% stress level until all had failed. For complete data sets see Andrews and Herzberg [5]. Table 3 provides the statistical summary of the Kevalr data.

We choose W, KW, TKW, EKW and BKW models to analyze this data. The purpose of the study was to investigate improvement in data fitting based on the choice of the model. The estimated parameters and their corresponding errors are provided in Table 4.

The values of $-\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC and KS test statistic and corresponding p value are provided in Table 5.

Kevlar data could be well analyzed using Weibull distribution. We investigate whether a significant improvement can be achieved by inducting additional parameters in the Weibull distribution. The results in Table 5 indicates that TKW is more likely to provide better fit than any other models. Note that TKW has only 5 parameters but BKW has six parameters. So higher the number of parameters doesn't necessarily

Table 5 The AIC, CAIC, BIC, HQIC and K-S test statistic of the Kevlar data

Model	Statistics						
	$-\ell(., x)$	AIC	CAIC	BIC	HQIC	D_KS	p value
TKW	121.2328	252.4655	253.3227	264.1192	257.1229	0.0918	0.5135
EKW	122.1001	254.2002	255.0573	265.8538	258.8575	0.0977	0.4348
BKW	122.1702	256.3403	257.5577	270.3247	261.9292	0.0988	0.4217
KW	122.2091	252.4181	252.9815	261.7411	256.144	0.1001	0.4055
W	122.5247	249.0494	249.2138	253.7108	250.9123	0.10997	0.2949

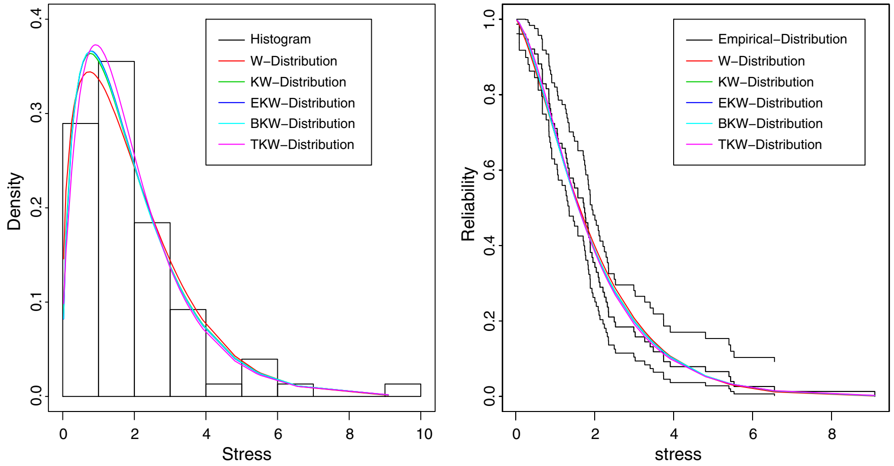

Fig. 3 Fitted pdf (left panel) and reliability plot (right panel) of Kevlar data

Table 6 Statistical summary of the phyrnax data

n	Minimum	Q_1	Median	Mean	Q_3	Maximum
142	11	205.8	325.5	388.9	525.8	1574

yield better model. Figure 3 displays the fitted density plot and the reliability plot for the Kevlar data using W, KW, TKW, BKW and EKW distributions. We can observe that TKW clearly captures both the pickedness and tail behavior of underlying data.

Example 2: Phyranx Data Modeling

Oropharyngeal cancer cells are found in the borders of the oropharynx. According to National Cancer Institute [39] these type of cancers are in most cases squamous cell carcinomas which are cancers arising from the surface cells of the throat. The pharynx data is taken from KalbfleischRoss [30]. The data was collected through a clinical trial to determine the extent to which the several covariates relate to subsequent survival in the treatment of carcinoma of the Oropharynx with 30% censoring. We used

Table 7 Estimated parameters and their standard errors (in parenthesis) for the pharynx data

Model	α	β	a	b	c	d	ν	λ
KBW	0.0026	1.1593	1.6464	1.2344	1.0058	1.1053	–	–
	(0.0002)	(0.7024)	(9.9589)	(3.7338)	(5.6398)	(4.6619)	–	–
TBW	0.0023	1.4340	1.2403	0.7601	–	–	–	0.7310
	(0.0002)	(0.3516)	(0.4800)	(0.3486)	–	–	–	(0.3550)
EBW	0.0039	1.2172	1.3389	0.7345	–	–	1.1723	–
	(0.0006)	(0.4766)	(6.6915)	(0.7666)	–	–	(6.2915)	–

Table 8 The AIC, CAIC, BIC, HQIC and Anderson–Darling, Cramér–Von Mises statistic of the pharynx data

Model	Statistics						
	$-\ell(.,x)$	AIC	CAIC	BIC	HQIC	$A*$	$W*$
TBW	969.87	1949.75	1950.20	1964.53	1955.76	0.2867	0.0304
KBW	970.37	1952.75	1953.37	1970.48	1959.96	0.3322	0.0344
EBW	970.38	1950.75	1951.19	1965.53	1956.76	0.3323	0.0343

survival time in days from the day of diagnosis to fit some compositional distributions and assess the order of compositions to find a preferable order. Table 6 provides the statistical summary of the phyranx data.

Due to the presence of incomplete beta function mathematical characterizations of generalized beta Weibull distribution using any generalizer is quite challenging. We use numerical methods to study the reliability behavior of generalized beta Weibull distributions using pharynx data. In particular Kumarswamy beta Weibull (KBW), transmuted beta Weibull (TBW) and exponentiated beta Weibull(EBW) distributions will be considered. The estimated parameters and their corresponding errors for fitted models are provided in Table 7.

The values of $-\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, Anderson–Darling, Cramér–Von Mises statistics are provided in Table 8. Observe that all three generalized beta Weibull distributions fit equally well to the subject data but among these the KBW is the least preferred due to higher number of parameters than EBW and TBW distributions. Based on the results Table 8 TBW is recommended as the best fitted model to analyze the Pharynx data.

Figure 4 displays the fitted density plot and the reliability plot for the phyranx data using TBW, KBW and EBW distributions.

7 Concluding Remarks

We analyzed the order of compositions of probability distributions with Weibull as a base distribution. A strong theoretical foundation and numerous successful real life applications of Weibull distribution are major driving factors for our choice of

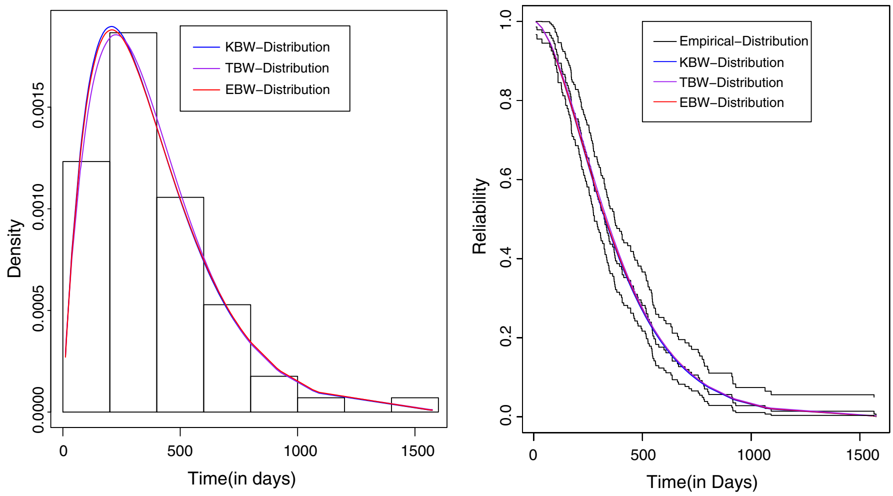

Fig. 4 Fitted pdf (left panel) and reliability plot (right panel) of phyranx data

base distribution. The Weibull distribution followed by other major compositions: beta, Kumaraswamy, exponentiated, and transmuted are assessed in the paradigm of theoretical strengths and real life applications. In a nutshell, additional number of parameters not necessarily help to achieve better fit. Nevertheless, we observed noticeable differences in reliability of the same composition for some specific values of the parameters. For Kevlar data, we used different means of model selection criterion. The statistics including AIC and BIC shows TBW with consistently better model with lower error and strongly aligning p values. The likelihood estimates of Weibull distribution followed by first and second compositions in different order taper off as more distributions are compounded. Similarly, for pharynx data we use three compositions with beta distribution followed by each of the beta, Kumaraswamy and transmuted generalizers. To the best of our knowledge this is the first attempt to generalize beta Weibull distribution using the genesis of other distributions. We did not find an universal directional alternative regarding these compositions because these kind of model are driven by the nature of the data. We are rather suggesting these types of compositions are local phenomenon. That is, a particular composition might not end up with better fit just because it possesses larger number of parameters. In summary, hierarchy of compositions are worth exploring for better fit of heavy tail distributions which can essentially capture the variations beyond the scope of classical alternatives.

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